Analytical Properties and Applications of Orthogonal Polynomials and Special Functions

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• Eta functions: These functions are a powerful tools for deriving the approximation of functions with trigonometric or hyperbolic variation which have oscillatory character.

• Eta-based function: The new set of based functions, the Eta-based function, has been introduced using the Eta functions. An essential property of the Eta-based functions is that they tend to the polynomial when the involved frequencies tend to zero. Thus, the Eta-based functions are suitable for attaining a good approximation of high oscillatory functions and polynomials.

• Orthogonal polynomials: These polynomials play the most important role in spectral methods and, therefore, it is necessary to highlight their relevant properties.

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The gamma function is used extension of the factorial function to complex numbers. The gamma function is defined for all complex numbers except the non-positive integers as:

Gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \mathrm{Re} z > 0. \tag{1}$$

By using integration by parts we find that

$$\Gamma(z+1) = z\Gamma(z), \quad \mathrm{Re} z > 0. \tag{2}$$

Further we have

$$\Gamma(1) = \int_0^\infty e^{-t} dt = - \left. e^{-t} \right|_0^\infty = 1. \tag{3}$$

Combining (2) and (3), this leads to

$$\Gamma(n+1) = n!. \tag{4}$$

The beta function $B(u, v)$ is also defined by means of an integral:

Beta function

$$B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}dt, \quad \mathrm{Re}u > 0, \; \mathrm{Re}v > 0. \tag{5}$$

The connection between the beta function and the gamma function is given by the following theorem:

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \quad \mathrm{Re}u > 0, \; \mathrm{Re}v > 0. \tag{6}$$

In order to prove this theorem we use the definition (1) to obtain

$$\Gamma(u)\Gamma(v) = \int_0^\infty e^{-t}t^{u-1}dt \int_0^\infty e^{-t}s^{v-1}ds = \int_0^\infty \int_0^\infty e^{-(t+s)}t^{u-1}\, s^{v-1}dt\, ds \tag{7}$$

Now we apply the change of variables $t = xy$ and $s = x(1-y)$ to this double integral. Note that $t + s = x$ and that $0 < t < \infty$ and $0 < s < \infty$ imply that $0 < x < \infty$ and $0 < y < 1$. The Jacobian of this transformation is

$$\frac{\partial(t,s)}{\partial(x,y)} = \begin{vmatrix} y & x \\ 1-y & -x \end{vmatrix} = -x.$$

Hence we have

$$\Gamma(u)\Gamma(v) = \int_0^1 \int_0^\infty e^{-x} x^{u-1}\, y^{u-1} x^{v-1}\,(1-y)^{v-1} x\,dx\,dy$$

$$= \int_0^\infty e^{-x} x^{u+v-1} dx \int_0^1 y^{u-1}(1-y)^{v-1} dy = \Gamma(u+v)\,B(u,v).$$

For real parameters $p_1, ..., p_\alpha$ and $q_1, ..., q_\beta$ $(q_j \neq 0, -1, -2, ...\ j = 1, ..., \beta)$, we define the generalized hypergeometric function ${}_\alpha F_\beta(p_1, ..., p_\alpha; q_1, ..., q_\beta; Y)$ according to

Generalized hypergeometric functions

$$ {}_\alpha F_\beta(p_1, ..., p_\alpha; q_1, ..., q_\beta; Y) = \sum_{k=0}^{\infty} \frac{(p_1)_k ... (p_\alpha)_k}{(q_1)_k ... (q_\beta)_k} \frac{Y^k}{k!}, \tag{8} $$

where $(p)_k$ is the Pochhammer symbol defined, in terms of the Gamma function $\Gamma(.)$, by

$$ (p)_0 = 1, \qquad (p)_k = p(p+1)(p+2)...(p+k-1) = \Gamma(p+k)/\Gamma(p), \quad k \in \mathbb{N}. \tag{9} $$

If $\alpha \leq \beta$, the series is absolutely convergent for all values of Y , if $\alpha = \beta + 1$, the series is convergent for $|Y| < 1$ and for $|Y| = 1$ the series is conditionally convergent. If $\alpha > \beta + 1$, the series is divergent.

The Bessel function of the first kind of real order μ has the series expansion as stated in

Bessel functions

$$J_\mu(Y) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\mu+1)}(\frac{Y}{2})^{2k+\mu}. \tag{10}$$

The infinite series in equation (10) will converge for all values of Y. The modified Bessel functions of the first kind are defined by

$$I_\mu(Y) = i^{-\mu} J_\mu(iY), \tag{11}$$

where $i = \sqrt{-1}$ is the complex unit. It is easy to show the modified Bessel functions of the first kind are a real function of Y.

Eta functions, denoted by $\eta_n(Y)$, $n > 0$ and $Y \neq 0$, are defined in terms of the recurrence relation (12)

Eta Functions

$$\eta_n(Y) = \frac{\eta_{n-2}(Y) - (2n-1)\eta_{n-1}(Y)}{Y}, \qquad n = 1, 2, 3, \ldots \tag{12}$$

where

$$\eta_{-1}(Y) = \begin{cases} \cos(|Y|^{\frac{1}{2}}) & Y \leq 0, \\ \cosh(Y^{\frac{1}{2}}) & Y > 0, \end{cases} \qquad \eta_0(Y) = \begin{cases} \frac{\sin(|Y|^{\frac{1}{2}})}{|Y|^{\frac{1}{2}}} & Y < 0, \\ 1 & Y = 0, \\ \frac{\sinh(Y^{\frac{1}{2}})}{Y^{\frac{1}{2}}} & Y > 0. \end{cases} \tag{13}$$

These functions have the following values at $Y = 0$:

$$\eta_n(0) = \frac{1}{(2n+1)!!}, \qquad n = 1, 2, ... \tag{14}$$

where !! is a double factorial. Eta functions have some well-known properties such as:

Series expansion:

$$\begin{aligned} \eta_n(Y) &= 2^n \sum_{k=0}^{\infty} \frac{(k+n)!}{(2k+2n+1)!} \frac{Y^k}{k!} \\ &= 2^{-(n+1)} \sqrt{\pi} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+n+\frac{3}{2})} \frac{\left(\frac{Y}{4}\right)^k}{k!}, \qquad n = 0, 1, ... \end{aligned} \tag{15}$$

Differentiation properties of Eta functions is defined as

Differentiation properties

$$\eta_n'(Y) = \frac{1}{2}\eta_{n+1}(Y), \quad n = -1, 0, 1, 2, ... \tag{16}$$

$\eta_n(Y),\ (n = 0, 1, 2, ...)$ is the suitably normalized regular solution of differential equation (17)

Differentiation properties

$$Yz'' + \frac{1}{2}(2n+3)z' - \frac{1}{4}z = 0. \tag{17}$$

Lemma

The Eta-functions can be defined by the Bessel functions as

$$\eta_n(-x^2) = \sqrt{\tfrac{\pi}{2}} x^{-(n+\frac{1}{2})} J_{n+\frac{1}{2}}(x)$$
$$\eta_n(+x^2) = \sqrt{\tfrac{\pi}{2}} x^{-(n+\frac{1}{2})} I_{n+\frac{1}{2}}(x)$$

(17)

Theorem 1: (Generating function)

The generating function of the Eta functions is obtained according to Eq. (18) as

$$\sqrt{\frac{\pi}{2}} e^{\frac{t}{2}+\frac{Y}{2t}} \times \frac{1}{\sqrt{t}} Erf \sqrt{\frac{t}{2}} = \sum_{n=0}^{\infty} \eta_n(Y) t^n, \tag{18}$$

where Erf, is an Error function (also called probability integral) as stated in Eq. (19)

Error function

$$Erf\,t = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} ds = \frac{2}{\sqrt{\pi}} e^{-t^2} \sum_{\gamma=0}^{\infty} \frac{2^{\gamma} t^{2\gamma+1}}{(2\gamma+1)!!} \,. \tag{19}$$

Proof: Using the Taylor series of $e^{\frac{Y}{2t}}$ and Eq. (19) we have

$$\sqrt{\frac{\pi}{2}}e^{\frac{t}{2}+\frac{Y}{2t}} \times \frac{1}{\sqrt{t}}Erf\sqrt{\frac{t}{2}} = \left(e^{\frac{Y}{2t}}\right) \times \left(\sqrt{\frac{\pi}{2}}e^{\frac{t}{2}} \times \frac{1}{\sqrt{t}}Erf\sqrt{\frac{t}{2}}\right)$$

$$= \left(\sum_{\nu=0}^{\infty} \frac{\left(\frac{Y}{2t}\right)^{\nu}}{\nu!}\right)\left(\sqrt{\frac{\pi}{2}}e^{\frac{t}{2}} \times \frac{1}{\sqrt{t}} \times \frac{2}{\sqrt{\pi}}e^{-\frac{t}{2}} \sum_{\gamma=0}^{\infty} \frac{2^{\gamma}\left(\frac{t}{2}\right)^{\frac{2\gamma+1}{2}}}{(2\gamma+1)!!}\right)$$

$$= \sum_{\nu=0}^{\infty} \frac{\left(\frac{Y}{2t}\right)^{\nu}}{\nu!} \sum_{\gamma=0}^{\infty} \frac{(t)^{\gamma}}{(2\gamma+1)!!}$$

$$= \sum_{\nu=0}^{\infty} \frac{\left(\frac{Y}{2t}\right)^{\nu}}{\nu!} \sum_{\gamma=0}^{\infty} \frac{\gamma!(2t)^{\gamma}}{(2\gamma+1)!} = \sum_{\nu,\gamma=0}^{\infty} \frac{2^{\gamma-\nu}\gamma!}{(2\gamma+1)!\nu!}Y^{\nu}t^{\gamma-\nu}, \tag{20}$$

now we want to pick out the coefficient of t^n in this expansion. For a fixed value of γ the coefficient of t^n is obtained by taking $\gamma - \nu = n$, *i.e.*, $\gamma = \nu + n$. Thus, for this special value of γ in Eq. (20), the coefficient of t^n can be obtained from the following relation

$$\frac{2^n \gamma!}{(2\gamma+1)!(\gamma-n)!} Y^{\gamma-n} = \text{the coefficient of } t^n. \tag{21}$$

The total coefficient of t^n in Eq. (20) is obtained by summing over all allowed values of γ. Since $\nu = \gamma - n$ and $\nu \geq 0$, we should have $\gamma \geq n$ so using Eq. (21), the total coefficient of t^n will be as

$$\sum_{\gamma=n}^{\infty} \frac{2^n \gamma!}{(2\gamma+1)!(\gamma-n)!} Y^{\gamma-n} = \sum_{k=0}^{\infty} \frac{2^n (n+k)!}{(2n+2k+1)!} \frac{Y^k}{k!} = \eta_n(Y), \tag{22}$$

where we have set $k = \gamma - n$.

Theorem 2. (Integral representation)

Eta functions of order n can be represented by the integral Eq. (23) as

$$\eta_n(Y) = \frac{2^{-(n+1)}\sqrt{\pi}}{2\pi i}\int_{-\infty}^{0^+} t^{-(n+\frac{3}{2})} e^{t+\frac{Y}{4t}}\, dt. \tag{23}$$

Proof: We have

$$\int_{-\infty}^{0^+} t^{-(n+\frac{3}{2})} e^t \times e^{\frac{Y}{4t}} dt = \int_{-\infty}^{0^+} t^{-(n+\frac{3}{2})} e^t \sum_{k=0}^{\infty} \frac{\left(\frac{Y}{4t}\right)^k}{k!} dt$$

$$= \int_{-\infty}^{0^+} t^{-(n+\frac{3}{2})} e^t \sum_{k=0}^{\infty} (4t)^{-k} \frac{Y^k}{k!} dt = \sum_{k=0}^{\infty} \frac{\left(\frac{Y}{4}\right)^k}{k!} \int_{-\infty}^{0^+} t^{-(k+n+\frac{3}{2})} e^t dt, \tag{24}$$

we also recall the Hankel's representation conforming to Eq. (25),

$$\int_{-\infty}^{0^+} t^{-(k+n+\frac{3}{2})} e^t dt = \frac{2\pi i}{\Gamma(k+n+\frac{3}{2})}, \tag{25}$$

substituting Eq. (25) into (24) leads to

$$\int_{-\infty}^{0^+} t^{-(n+\frac{3}{2})} e^t \times e^{\frac{Y}{4t}} dt = \sum_{k=0}^{\infty} \frac{2\pi i}{\Gamma(k+n+\frac{3}{2})} \frac{\left(\frac{Y}{4}\right)^k}{k!}, \tag{26}$$

consequently, from Eqs. (15) and (26), we obtain

$$\frac{2^{-(n+1)}\sqrt{\pi}}{2\pi i} \int_{-\infty}^{0^+} t^{-(n+\frac{3}{2})} e^{t+\frac{Y}{4t}} dt = 2^{-(n+1)}\sqrt{\pi} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+n+\frac{3}{2})} . \frac{\left(\frac{Y}{4}\right)^k}{k!} = \eta_n(\tag{27}$$

Theorem 3. (Laplace transform)

The Laplace transform of Eta functions is expressed following Eq. (28)

$$\mathcal{L}\{\eta_n(Y); s\} = \frac{2^{-(n+1)}\sqrt{\pi}}{s} E_{1,n+\frac{3}{2}}\left(\frac{1}{4s}\right). \tag{28}$$

Proof: Using Eq. (15) we have

$$\mathcal{L}\{\eta_n(Y);s\} = \int_0^\infty e^{-sY}\eta_n(Y)dY = \int_0^\infty e^{-sY}2^{-(n+1)}\sqrt{\pi}\sum_{k=0}^{\infty}\frac{(\frac{Y}{4})^k}{k!\Gamma(k+n+\frac{3}{2})}dY$$

$$= 2^{-(n+1)}\sqrt{\pi}\sum_{k=0}^{\infty}\frac{1}{4^k k!\Gamma(k+n+\frac{3}{2})}\int_0^\infty e^{-sY}Y^k dY$$

$$= \frac{2^{-(n+1)}\sqrt{\pi}}{s}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4s}\right)^k}{\Gamma(k+n+\frac{3}{2})} = \frac{2^{-(n+1)}\sqrt{\pi}}{s}E_{1,n+\frac{3}{2}}\left(\frac{1}{4s}\right), \tag{29}$$

where

$$E_{\alpha,\beta}(Y) = \sum_{k=0}^{\infty}\frac{Y^k}{\Gamma(\alpha k+\beta)}, \quad \alpha > 0,\ \beta \in \mathbb{C}, \tag{30}$$

is the generalized Mittag-Leffler function

Lemma

The product of two Eta-based functions $\eta_n(Y)\eta_m(Y)$ can be obtained as reported by

$$\eta_n(Y)\eta_m(Y) =$$

$$\frac{\pi 2^{-n-m-2}}{\Gamma(n+\frac{3}{2})\Gamma(m+\frac{3}{2})} {}_2F_3(\tfrac{2+n+m}{2}, \tfrac{3+n+m}{2}; \tfrac{3}{2}+m, \tfrac{3}{2}+n, 2+n+m; x)$$

Theorem 4.

The Eta functions satisfy the following relations: (28)

$$\eta_n(Z) = \eta_n(0) + Z\, D_n(Z), \quad n = -1, 0, 1, 2, 3, ..., \tag{31}$$

where

$$D_n(Z) = \eta_n(0)\left[\frac{1}{2}\eta_0^2(\frac{Z}{4}) - \sum_{i=0}^{n+1}(2i-3)!!\eta_i(Z)\right]. \tag{32}$$

Proof: At first observe that, from definition

$$\eta_n(Z) = \frac{\eta_{n-2}(Z) - (2n-1)\eta_{n-1}(Z)}{Z}, \qquad n = 0, 1, 2, ...,$$

we have

$$\eta_n(Z) = \frac{[\eta_{n-1}(Z) - Z\eta_{n+1}(Z)]}{2n+1}, \qquad n = 1, 2, 3, ..., \tag{33}$$

and proceed by induction on n.

$$\eta_0(Z) = \begin{cases} \frac{1}{2Z}(e^{\sqrt{Z}} - e^{-\sqrt{Z}}) & Z \neq 0, \\ 1 & Z = 0, \end{cases}$$

$$\eta_0(\tfrac{Z}{4}) = \tfrac{1}{\sqrt{Z}}(e^{\frac{\sqrt{Z}}{2}} - e^{-\frac{\sqrt{Z}}{2}}) \Rightarrow Z\,\eta_0^2(\tfrac{Z}{4}) = (e^Z + e^{-Z} - 2) = \tfrac{1}{2}\eta_{-1}(Z) - 1,$$

so we have

$$\eta_{-1}(Z) = 1 + 2Z\,\eta_0^2(\frac{Z}{4}) = \eta_{-1}(0) + D_{-1}(Z),$$

Let us suppose $n \geq 0$ and let Eq. (31) and Eq. (32) be valid for $n-1$, i.e.

$$\eta_{n-1}(Z) = \eta_{n-1}(0) + Z\, D_{n-1}(Z), \tag{34}$$

$$D_{n-1}(Z) = \eta_{n-1}(0)\left[\frac{1}{2}\eta_0^2(\frac{Z}{4}) - \sum_{i=0}^{n}(2i-3)!!\eta_i(Z)\right]. \tag{35}$$

By substituting Eq. (34) in Eq. (33), and by using $\eta_n(0) = \frac{1}{(2n+1)!!}$, which shows that $\eta_n(0) = \eta_{n-1}(0)/(2n+1)$, we have

$$\eta_n(Z) = \frac{\eta_{n-1}(0) + Z\,(D_{n-1}(Z) - \eta_{n+1}(Z))}{2n+1} = \eta_n(0) + Z\frac{D_{n-1}(Z) - \eta_{n+1}(Z}{2n+1}$$

From Eq. (35) we have

$$\frac{\frac{D_{n-1}(Z)-\eta_{n+1}(Z)}{2n+1}}{} = \frac{\eta_{n-1}(0)\left[\frac{1}{2}\eta_0^2(\frac{Z}{4}) - \sum_{i=0}^{n+1}(2i-3)!!\eta_i(Z) - (2n-1)!!\,\eta_{n+1}(Z)\right]}{2n+1} = D_n(Z).$$

Theorem 5.

For $n = -1, 0, 1, ...$ we have

$$[Z^n \eta_s(aZ)]^{(m)} = \frac{1}{2^m} \sum_{j=0}^{J} 2^j \left(\begin{array}{c} m \\ j \end{array} \right) \frac{n!}{(n-j)!} Z^{n-j} a^{m-j} \eta_{m-j+s}(aZ), \tag{36}$$

where $J = minm, n$.

Proof: On using the Leibniz formula for the product $f(Z)g(Z)$

$$[f(Z).g(Z)]^{(m)} = \sum_{j=0}^{m} \left(\begin{array}{c} m \\ j \end{array} \right) f^{(j)}(Z).g^{(m-j)}(Z),$$

for $f(Z) = Z^n$ and $g(Z) = \eta_s(aZ)$ and the relations

$$[Z^n]^{(j)} = \begin{cases} \frac{n!}{(n-j)!} Z^{n-j} & j \leq n \\ 0 & j > n \end{cases} \quad and \quad [\eta_s(aZ)]^{(i)} = \frac{1}{2^i} a^i \eta_{i+s}(aZ),$$

the stated relation results directly

Eta-based functions

Eta-based functions, denoted by $\varphi_n(t)$ are defined according to Eq. (37) as

$$\varphi_n(t) = t^{n-1}\eta_{\lfloor \frac{n}{2} \rfloor -1}(Y(t)),\ n = 1, 2, \cdots \tag{37}$$

where $Y(t) = -\xi^2 t^2$ in the trigonometric case and $Y(t) = \xi^2 t^2$ in the hyperbolic case. These functions have the following properties

$$\varphi_{n+1}(t) = t\varphi_n(t), \quad \text{for even number } n \geq 2. \tag{38}$$

$$\varphi_{n+2}(t) = \frac{t\varphi_{n-1}(t) - (n-1)\varphi_n(t)}{\mp\xi^2}, \quad \text{for even number } n \geq 2, \quad \xi \neq 0, \tag{39}$$

where the upper/lower sign is for oscillatory/hyperbolic case. An essential property of the Eta-based functions is that they tend to the classical power function (or polynomial) when $\xi = 0$

Theorem

The Eta-based functions $\varphi_n(t)$ can be defined by the Bessel functions as stated in Eq. (40)

$$\varphi_n(t) = \begin{cases} \sqrt{\frac{\pi}{2}}\xi^{\frac{1}{2}-\lfloor\frac{n}{2}\rfloor}t^{n-\lfloor\frac{n}{2}\rfloor-\frac{1}{2}}J_{\lfloor\frac{n}{2}\rfloor-\frac{1}{2}}(\xi t), & Y = -\xi^2t^2, \\ \\ \sqrt{\frac{\pi}{2}}\xi^{\frac{1}{2}-\lfloor\frac{n}{2}\rfloor}t^{n-\lfloor\frac{n}{2}\rfloor-\frac{1}{2}}I_{\lfloor\frac{n}{2}\rfloor-\frac{1}{2}}(\xi t), & Y = \xi^2t^2. \end{cases} \tag{40}$$

Proof: From the series expansion of Eta functions, the definition of Eta-Based function and using the Legendre duplication formula (**??**), we have

Theorem

For $\xi \neq 0$, the product of two Eta-based functions $\varphi_n(t)\varphi_m(t)$ can be obtained as reported by Eq. (41)

$$\varphi_n(t)\varphi_m(t) = \pi 2^{-\left(\lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor\right)} t^{n+m-2} \sum_{k=0}^{\infty} \frac{\binom{2k+\lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor - 1}{k}}{\Gamma(k+\lfloor \frac{n}{2} \rfloor - \frac{1}{2})\Gamma(k+\lfloor \frac{m}{2} \rfloor - \frac{1}{2})} \left(\frac{\mp\xi^2 t^2}{4}\right)^k$$

$$= \pi 2^{-\left(\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor\right)} t^{n+m-2} \frac{{}_2F_3\left(\frac{1}{2}\lfloor \frac{n}{2} \rfloor + \frac{1}{2}\lfloor \frac{m}{2} \rfloor, \frac{1}{2} + \frac{1}{2}\lfloor \frac{n}{2} \rfloor + \frac{1}{2}\lfloor \frac{m}{2} \rfloor; \frac{1}{2} + \lfloor \frac{m}{2} \rfloor, \frac{1}{2} + \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor; \mp\xi^2 t^2\right)}{\Gamma(\lfloor \frac{n}{2} \rfloor + \frac{1}{2})\Gamma(\lfloor \frac{m}{2} \rfloor + \frac{1}{2})}. \tag{41}$$

Best approximation and operational matrices:
Suppose $f(t) \in L^2[0,1]$ and

$$f_N^H(t) = H^T A = a_1 h_1(t) + a_2 h_2(t) + \cdots + a_N h_N(t), \tag{42}$$

is the best approximation to f out of H where

$$H(t) = [h_1(t),\, h_2(t), \ldots, h_N(t)]^T, \qquad A = [a_1, a_2, \ldots, a_N]^T, \tag{43}$$

are the base functions and coefficients vector. We have two next theorems if we choose the Eta-based functions as basis functions in Eq. (42).

Theorem (Operational matrix of derivative)

The derivative of the $H(t) = [\varphi_1(t), \varphi_2(t), \ldots, \varphi_N(t)]^T$ where $\varphi_i(t)$ defined in Eq. (37) satisfies the following relation

$$H'(t) = D(t)H(t), \tag{44}$$

where $D(t) = [d_{ij}]_{N\times N}$ is the operational matrix of derivative.

$$D(t) = \begin{cases} D_1(t) & \text{If } N \text{ is even,} \\ \\ D_2(t) & \text{If } N \text{ is odd,} \end{cases} \tag{45}$$

$$
D_1 = \begin{bmatrix}
0 & \mp\xi^2 & & & & & & \\
1 & 0 & 0 & & & & & \\
t & 1 & 0 & 0 & & & & \\
 & 0 & 1 & 0 & 0 & & & \\
 & & t & 1 & 0 & 0 & & \\
 & & & \ddots & \ddots & \ddots & \ddots & \\
 & & & & t & 1 & 0 & 0 \\
 & & & & & 0 & 1 & 0
\end{bmatrix}, D_2 = \begin{bmatrix}
0 & \mp\xi^2 & & & & & & \\
1 & 0 & 0 & & & & & \\
t & 1 & 0 & 0 & & & & \\
 & 0 & 1 & 0 & 0 & & & \\
 & & t & 1 & 0 & 0 & & \\
 & & & \ddots & \ddots & \ddots & \ddots & \\
 & & & & 0 & 1 & 0 & 0 \\
 & & & & & t & 1 & 0
\end{bmatrix}.
$$

For $\xi = 0$ we have:

$$D(t) = \begin{cases} D_3 & \text{If } N \text{ is even,} \\ \\ D_4 & \text{If } N \text{ is odd,} \end{cases} \tag{46}$$

where

$$D_3 = \begin{bmatrix} 0 & 0 & & & & & & & \\ 1 & 0 & 0 & & & & & & \\ & 2 & 0 & 0 & & & & & \\ & & 1 & 0 & 0 & & & & \\ & & & 4 & 0 & 0 & & & \\ & & & & \ddots & \ddots & \ddots & & \\ & & & & & 1 & 0 & 0 & \\ & & & & & & N-2 & 0 & 0 \\ & & & & & & & 1 & 0 \end{bmatrix}, D_4 = \begin{bmatrix} 0 & 0 & & & & & & \\ 1 & 0 & 0 & & & & & \\ & 2 & 0 & 0 & & & & \\ & & 1 & 0 & 0 & & & \\ & & & 4 & 0 & 0 & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & & N-3 & 0 & 0 \\ & & & & & & 1 & 0 \\ & & & & & & & N-1 \end{bmatrix}$$

Theorem (Dual operational matrix)

The dual operational matrix of the $H(t) = [\varphi_1(t), \varphi_2(t), \ldots, \varphi_N(t)]^T$ can be obtained according to Eq. (47) as

$$\int_0^1 H(t)H^T(t)dt = Q_H, \tag{47}$$

where Q_H is the $N \times N$ dual operational matrix and

$$Q_H = \begin{bmatrix} \phi(1,1) & \phi(1,2) & \cdots & \phi(1,N) \\ \phi(2,1) & \phi(2,2) & \cdots & \phi(2,N) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(N,1) & \phi(N,2) & \cdots & \phi(N,N) \end{bmatrix}, \tag{48}$$

in which

$$
\phi(n,m)=\begin{cases} \dfrac{\pi 2^{-\left(\left\lfloor \frac{n}{2}\right\rfloor+\left\lfloor \frac{m}{2}\right\rfloor\right)}}{\Gamma(\left\lfloor \frac{n}{2}\right\rfloor+\frac{1}{2})\Gamma(\left\lfloor \frac{m}{2}\right\rfloor+\frac{1}{2})} \displaystyle\sum_{k=0}^{\infty} \dfrac{(\mp\xi^2)^k(\frac{1}{2}\left\lfloor \frac{n}{2}\right\rfloor+\frac{1}{2}\left\lfloor \frac{m}{2}\right\rfloor)_k(\frac{1}{2}+\frac{1}{2}\left\lfloor \frac{n}{2}\right\rfloor+\frac{1}{2}\left\lfloor \frac{m}{2}\right\rfloor)_k(\frac{n+m-1}{2})_k}{(n+m-1)k!(\frac{1}{2}+\left\lfloor \frac{m}{2}\right\rfloor)_k(\frac{1}{2}+\left\lfloor \frac{n}{2}\right\rfloor)_k(\left\lfloor \frac{n}{2}\right\rfloor+\left\lfloor \frac{m}{2}\right\rfloor)_k(\frac{n+m+1}{2})_k} & \text{If } \xi\neq 0 \\ \dfrac{1}{(n+m-1)!(2\left\lfloor \frac{m}{2}\right\rfloor-1)!!(2\left\lfloor \frac{n}{2}\right\rfloor-1)!!} & \text{If } \xi=0 \end{cases} \tag{49}
$$

Proof: Since

$$
\int_0^1 H(t)H^T(t)dt=\begin{bmatrix} \int_0^1 \varphi_1(t)\varphi_1(t)dt & \int_0^1 \varphi_1(t)\varphi_2(t)dt & \cdots & \int_0^1 \varphi_1(t)\varphi_N(t)dt \\ \int_0^1 \varphi_2(t)\varphi_1(t)dt & \int_0^1 \varphi_2(t)\varphi_2(t)dt & \cdots & \int_0^1 \varphi_2(t)\varphi_N(t)dt \\ \vdots & \vdots & \ddots & \vdots \\ \int_0^1 \varphi_N(t)\varphi_1(t)dt & \int_0^1 \varphi_N(t)\varphi_2(t)dt & \cdots & \int_0^1 \varphi_N(t)\varphi_N(t)dt \end{bmatrix},
$$

we have

$$\phi(i,j) = \int_0^1 \varphi_i(t)\varphi_j(t)dt, \tag{50}$$

by integrating of the product of two Eta-based functions given in Eq. (41) on $[0,1]$, the result is obtained directly for $\xi \neq 0$. For $\xi = 0$, the result can be obtained by using Eqs. (14) and (37).

State-dependent and time-dependent neutral delay equation

In this section, we use the Eta-based function to develop the new numerical method for the state-dependent and time-dependent neutral delay equation as stated in Eq. (51)

$$\begin{cases} x'(t) = g(t, x(t), x(t - \Theta_1(t, x(t))), x'(t - \Theta_2(t, x(t)))), \\ x(0) = x_0, \qquad\qquad 0 \leq t \leq 1. \end{cases} \tag{51}$$

In Eq. (51),

$$x(t) = [x_1(t), x_2(t), \cdots, x_\rho(t)]^T \in \mathbb{R}^\rho, \tag{52}$$

is a real-valued ρ-vector function and

$$g(t) = [g_1(t), g_2(t), \cdots, g_\rho(t)]^T, \tag{53}$$

is assumed to be a sufficiently smooth real-valued ρ-vector function. Also, Θ_1, Θ_2 are assumed to be continuous functions for all $t \in [0, 1]$.

Numerical method

This section is devoted to presenting a new numerical method for solving the problem given in Eq. (51). Using Eq. (42) the best approximation of $x_i(t)$, $i = 1, 2, \cdots, \rho$ is

$$x_i(t) = H^T(t)A_i, \tag{54}$$

and

$$x(t) = \hat{H}(t)\hat{A}, \tag{55}$$

where $\hat{A}$ is a $\rho N \times 1$ vector given by

$$\hat{A} = [A_1, A_2, ..., A_\rho]^T, \tag{56}$$

and

$$\hat{H}(t) = I_\rho \otimes H^T(t), \tag{57}$$

in which I_ρ is the ρ dimensional identity matrix, $\hat{H}(t)$ is $\rho \times \rho N$ matrix as well, and $\otimes$ denotes Kronecker product. Using Eq. (54) and Theorem 2.6, we have

$$x_i'(t) = H^T(t) D^T(t) A_i. \tag{58}$$

Using Eqs. (52) and (54) we get

$$x'(t) = \hat{H}(t)\hat{D}(t)\hat{A}, \tag{59}$$

where $\hat{D}(t)$ is $\rho N \times \rho N$ matrix as

$$\hat{D}(t) = I_\rho \otimes D^T(t).$$

Substituting Eqs. (55) and (59) into (51), we have

$$\hat{H}(t)\hat{D}(t)\hat{A} =$$

$$g(t, \hat{H}(t)\hat{A}, \hat{H}(t - \Theta_1(t, \hat{H}(t)\hat{A}))\hat{A}, \hat{H}(t - \Theta_2(t, \hat{H}(t)\hat{A}))\hat{D}(t - \Theta_2(t, \hat{H}(t)\hat{A}))\hat{A}). \tag{60}$$

Next we collocate Eq. (60) at the Chebyshev nodes in $[0, 1]$

$$t_j = \frac{1}{2}\cos\frac{\pi(2j+1)}{2(N+1)} + \frac{1}{2}, \quad j = 0, 1, ..., N-1, \tag{61}$$

to obtain a system of ρN nonlinear equations as

$$W = \hat{H}(t_j)\hat{D}(t_j)\hat{A} - g(t_j, \hat{H}(t_j)\hat{A}, \hat{H}(t_j - \Theta_1(t_j, \hat{H}(t_j)\hat{A}))\hat{A}, \hat{H}(t_j - \Theta_2(t_j, \hat{H}(t_j)\hat{A}))\hat{D}(t_j - \Theta_2(t_j, \hat{H}(t_j)\hat{A}))\hat{A}) = 0. \tag{62}$$

Similar to Eq. (55), corresponding matrix form for the initial condition $x(0) = x_0$ is according to Eq. (63)

$$V = \hat{H}(0)\hat{A} - x_0 = 0. \tag{63}$$

Replacing V instead of the ρ last row of W, we have a set of ρN nonlinear equations which can be solved for the elements of $\hat{A}$ using the well Newton's iterative method. Finally, we calculate $x(t)$ given in Eq. (55).

Error estimate

This section aims to estimate the error norm for the numerical method. For ease of exposition but without any loss of generality, we describe convergence analysis for $\rho = 1$ and $x_1 = x$. At first, we suppose that $H^{\mu}(0,1)$ with $\mu \geq 0$ is a Sobolev space equipped with the norm

$$\| x \|_{H^{\mu}(0,1)} = \left(\sum_{j=0}^{\mu} \int_0^1 |x^{(j)}(t)|^2 w(t) dt \right)^{\frac{1}{2}} = \left(\sum_{j=0}^{\mu} \| x^{(j)} \|_{L^2(0,1)}^2 \right)^{\frac{1}{2}}. \tag{64}$$

To continue the error discussion, the following Theorem is recalled.

Theorem

Assume that x be a member of Sobolev space $H^{\mu}(0,1)$ with $\mu \geq 0$, and $P_n(2t-1)$ be the well-known shifted Legendre polynomials defined on the interval $[0,1]$. Let $\sum\limits_{n=0}^{N} \mathbf{a}_n P_n(2t-1) \in \Pi_N$, denotes the best approximation of x using the set of shifted Legendre polynomials, where Π_N is the space of all polynomials of degree less than or equal to N. Then we have

$$\left\| x - \sum_{n=0}^{N} \mathbf{a}_n P_n(2t-1) \right\|_{L^2(0,1)} \leq cN^{-\mu} |x|_{H^{\mu;N}(0,1)}, \tag{65}$$

where c is a constant positive independent of N and x and

$$|x|_{H^{\mu;N}(0,1)} = \left(\sum_{i=\min\{\mu, N+1\}}^{\mu} \left\| x^{(i)} \right\|_2^2 \right)^{\frac{1}{2}}. \tag{66}$$

Theorem

Suppose that x be a member of Sobolev space $H^{\mu}(0,1)$ with $\mu \geq 0$, and φ_n be the Eta-based functions defined on the interval $[0,1]$. Assume that $x_N(t) = \sum_{n=1}^{N} a_n \varphi_n(t)$ denotes the approximation of x using the set of Eta-based functions. Then we have

$$\left\| x - \sum_{n=1}^{N} a_n \varphi_n(t) \right\|_{L^2(0,1)} \leq \; cN^{-\mu} |x|_{H^{\mu;N}(0,1)} + \sum_{n=1}^{N} \sqrt{\pi} 2^{-\lfloor \frac{n}{2} \rfloor} \varepsilon |a_n|. \quad (67)$$

Theorem

Let $x \in H^{\mu}(0,1)$ be the exact solution of Eq. (51) and $\bar{x}_N = H^T \bar{A} = \sum_{n=1}^{N} \bar{a}_n \varphi_n(t)$ be the approximate solution of this equation obtained by the proposed method. Then, we have

$$\|x - \bar{x}_N\|_{L^2(0,1)} \leq cN^{-\mu} |x|_{H^{\mu;N}(0,1)} + \sum_{n=1}^{N} \sqrt{\pi} 2^{-\lfloor \frac{n}{2} \rfloor} \varepsilon |a_n|$$

$$+ \|A - \bar{A}\|_2 \left(\sum_{n=1}^{N} \frac{\pi 2^{-2[\frac{n}{2}]}}{(-1+2n)\{\Gamma(\lfloor \frac{n}{2} \rfloor + \frac{1}{2})\}^2} {}_2F_3 \left(n - \frac{1}{2}, \lfloor \frac{n}{2} \rfloor; n + \frac{1}{2}, 2\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + \frac{1}{2}; \right. \right. \tag{68}$$

Numerical example

In this section, we assess the new numerical method to derive the numerical solution of Eq.(51) for different cases. We consider different formats of the delay term, including a pantograph delay where the delay term is represented as $x(qt)$, and a time-dependent delay where the delay term is expressed as $x(\tau(t))$, and a state-dependent delay where a delay term is introduced as $x(t - \Theta(t, x(t)))$. In each example, we present the absolute error for each case to compare the results.

Case 1: We choose Eta-based functions as a base. In this case

$$H(t) = [\varphi_1(t), \varphi_2(t), \ldots, \varphi_N(t)]^T \tag{69}$$

is defined on $t \in [0, 1]$ where $\varphi_i(t)$ has been introduced in Eq. (37).

Case 2: We choose Legendre polynomials as a base. Legendre polynomials, $P_n(t)$, is defined on the interval $(-1, 1)$ using the following recursive formula

$$P_n(t) = 2P_{n-1}(t) - P_{n-2}(t), \ n = 2, 3, ..., \tag{70}$$

where $P_0(t) = 1$ and $P_1(t) = t$. These polynomials are orthogonal with respect to the weight $w(t) = 1$ on the interval $[-1, 1]$. In this case

$$H(t) = [P_0(2t-1),\ P_1(2t-1), \ldots, P_{N-1}(2t-1)]^T \tag{71}$$

is defined for $t \in [0, 1]$.

Case 3: We choose $H(t) = [\psi_0(t), \psi_1(t), \dots, \psi_{N-1}(t)]^T$ as a base where

$$\psi_i(t) = \begin{cases} cos(i \times t), & \text{if i is even} \\ sin(i \times t), & \text{if i is odd} \end{cases}$$

and $t \in [0, 1]$. In some specific cases we could consider only $sin(i \times t)$ or $cos(i \times t)$ as the base.

Example 1. Pantograph delay differential equation

In this example, we consider Eq. (51) where $\rho = 1$ and $g(t) = x(\frac{t}{2}) + \frac{e^t(t+1)}{2} + \frac{e^{-t}(t-1)}{2} - \frac{t}{2}\sinh\frac{t}{2}$. In this case, we have a delay differential equation of pantograph type with an exact solution $x(t) = tsinh(t)$. In this case, for reaching the absolute error of order $\mathcal{O}(10^{-16})$, the CPU time taken in Legendre polynomials was almost 42 times greater than that in Eta-based functions, and this accuracy was not achieved when we used trigonometric functions. The absolute error is presented in Table 1. In this table, we choose four first terms of the base for all three different choices of base functions.

Table: Absolute error for example 1.

t	Eta-based N=4	Legendre N=4	Trigonometric N=4
0.2	3.60×10^{-16}	2.05×10^{-2}	5.05×10^{-2}
0.4	5.27×10^{-16}	2.50×10^{-2}	6.16×10^{-2}
0.6	6.10×10^{-16}	2.67×10^{-2}	6.47×10^{-2}
0.8	6.66×10^{-16}	3.17×10^{-2}	7.71×10^{-2}

Example 2. Multi pantograph delay differential equation

In this example, we consider the multi-pantograph delay differential equation. We assume, in Eq. (51), $\rho = 1$ and $g(t) = -x(t) - e^{-\frac{t}{2}}\sin(\frac{t}{2})x(\frac{t}{2}) - 2e^{-\frac{3t}{4}}\cos(\frac{t}{2})\sin(\frac{t}{4})x(\frac{t}{4})$. The exact solution is $x(t) = e^{-t}cos(t)$. The absolute error is presented in Table 4. Also, CPU time used (in seconds) for different values of N is given in Table 2.

Table: Absolute error for example 2.

t	Eta-based functions			Trigonometric functions		
	$N=3$	$N=7$	$N=11$	$N=3$	$N=7$	$N=11$
0.2	1.2×10^{-2}	1.2×10^{-5}	1.6×10^{-10}	5.2×10^{-3}	6.9×10^{-4}	4.2×10^{-5}
0.4	1.5×10^{-2}	9.5×10^{-6}	1.2×10^{-10}	7.5×10^{-3}	5.3×10^{-4}	3.0×10^{-5}
0.6	1.4×10^{-2}	7.1×10^{-6}	9.1×10^{-11}	5.8×10^{-3}	4.0×10^{-4}	2.2×10^{-5}
0.8	1.0×10^{-2}	4.9×10^{-6}	6.3×10^{-11}	2.3×10^{-3}	2.6×10^{-4}	1.6×10^{-6}

Table: CPU time used corresponding to Eta-based functions for solving example 2.

Absolute error	$\mathcal{O}(10^{-2})$	$\mathcal{O}(10^{-3})$	$\mathcal{O}(10^{-5})$	$\mathcal{O}(10^{-10})$
CPU time	$(N=3)$ 0.001	$(N=5)$ 0.016	$(N=7)$ 0.031	$(N=11)$ 0.157

Example 3. Time-dependent neutral delay differential equation

To examine the effectiveness of the proposed method for time-dependent neutral delay differential equations, we consider Eq. (51) with $\rho = 1$ and $g(t) = -x(\Theta(t)) + x'(\Theta(t)) + cosh(t) - \frac{1}{t+1}$. Also, we assume $\Theta(t) = ln(t+1)$. The exact solution is chosen as $x(t) = sinh(t)$. Table 3 shows the absolute error for this case. In this example, reaching the absolute error of order $\mathcal{O}(10^{-16})$, the CPU time taken in Legendre polynomials was almost 66 times greater than that in Eta-based functions, and the CPU time taken in trigonometric functions was nearly 22 times greater than that in Eta-based functions.

Table: Absolute error for example 3.

t	Eta-based functions $N = 3$	Legendre polynomials $N = 3$	Trigonometric functions $N = 3$
0.2	1.3877×10^{-16}	4.0251×10^{-2}	3.9415×10^{-2}
0.4	1.6653×10^{-16}	4.9112×10^{-2}	6.3251×10^{-2}
0.6	2.2204×10^{-16}	3.4987×10^{-2}	6.1952×10^{-2}
0.8	2.2204×10^{-16}	6.9415×10^{-3}	3.7040×10^{-2}

Integral equation

In this section, we develop a new numerical method based on the Eta functions for solving the system of Fredholm and Volterra integral equations

$$P(t) = G(t) + \lambda_1 \int_0^t K_1(t,s,P(s))ds + \lambda_2 \int_0^1 K_2(t,s,P(s))ds, \quad 0 \le t, s \le 1, \tag{72}$$

where

$$\begin{aligned}
&P(t) = [\rho_1(t), ..., \rho_n(t)]^T,\\
&G(t) = [g_1(t), ..., g_n(t)]^T,\\
&K_1(t,s,P(s)) = [\kappa_1^1(t,s,P(s)), ..., \kappa_n^1(t,s,P(s))]^T,\\
&K_2(t,s,P(s)) = [\kappa_1^2(t,s,P(s)), ..., \kappa_n^2(t,s,P(s))]^T
\end{aligned}$$

and λ_1 and λ_2 are constant vectors.

The best approximation of $\rho_i(t)$ in Eq. (72) is

$$\rho_i(t) = H^T(t) D_i, \tag{73}$$

and

$$P(t) = \hat{H}(t)\hat{D}, \tag{74}$$

where $\hat{D}$ is a $nN \times 1$ vector given by

$$\hat{D} = [D_1, D_2, ..., D_n]^T, \tag{75}$$

and

$$\hat{H}(t) = I_n \otimes H^T(t), \tag{76}$$

where I_n is the n dimensional identity matrix. Also, $\hat{H}(t)$ is $n \times nN$ matrix and $\otimes$ shows Kronecker product.

Replacing Eq. (74) in (72), we have

$$\hat{H}(t)\hat{D} = G(t) + \lambda_1 \int_0^t K_1(t, s, \hat{H}(s)\hat{D})ds + \lambda_2 \int_0^1 K_2(t, s, \hat{H}(s)\hat{D})ds, \tag{77}$$

Using the Gauss-Legendre numerical integration for evaluating the integral in Eq. (77), we get

$$\hat{H}(t)\hat{D} = G(t)+\lambda_1 \sum_{i=0}^{p} \omega_i K_1(t, \frac{t}{2}+\frac{t}{2}\gamma_i, \hat{H}(\frac{t}{2}+\frac{t}{2}\gamma_i)\hat{D})+\lambda_2 \sum_{i=0}^{p} \omega_i K_2(t, \frac{1}{2}+\frac{1}{2}\gamma_i, \hat{H}(\frac{1}{2}+\frac{1}{2}\gamma_i)\hat{D}), \quad 0 \leq t \leq 1, \tag{78}$$

where ω_i and γ_i are weights and nods of Gauss-Legendre. Using Eq. (78), we introduce the residual of the problem as

$$R(t,\hat{D}) = \hat{H}(t)\hat{D} - G(t) - \lambda_1 \sum_{i=0}^{p} \omega_i K_1(t, \tfrac{t}{2} + \tfrac{t}{2}\gamma_i, \hat{H}(\tfrac{t}{2} + \tfrac{t}{2}\gamma_i)\hat{D})$$

$$-\lambda_2 \sum_{i=0}^{p} \omega_i K_2(t, \tfrac{1}{2} + \tfrac{1}{2}\gamma_i, \hat{H}(\tfrac{1}{2} + \tfrac{1}{2}\gamma_i)\hat{D}),$$

and collocate this equation at the extreme points of the Chebyshev polynomial to get nN nonlinear equations which can be solved for the elements of $\hat{D}$. We use Newton's iterative method to solve the nonlinear equations for the elements of $\hat{D}$. Finally, we calculate $P(t)$ given in Eq. (74).

Example 4:

In this example, we assume $n = 1$, $K_2(t, s, \rho(s)) = k(t, s)e^{\rho(s)}$ where $k(t, s) = ts$ and the exact solution is $\rho(t) = cos(t)$. The absolute error is presented in Table 4. In this table, we choose four first terms of the base for all three different choices of base functions.

Table: Absolute error (Example 4)

t	Eta-based functions	Legendre polynomials	Trigonometric
0.2	3.8×10^{-11}	1.1×10^{-4}	3.5×10^{-11}
0.4	7.5×10^{-11}	8.6×10^{-5}	7.9×10^{-11}
0.6	1.1×10^{-10}	1.1×10^{-4}	1.1×10^{-10}
0.8	1.5×10^{-10}	1.2×10^{-5}	1.5×10^{-10}

Example 5: In this example, we assume
$n = 1, \quad K_2(t, s, \rho(s)) = k(t, s)\rho^2(s)$ where $k(t, s) = ts$ and the exact solution is $\rho(t) = tsinh(t)$. The absolute error is presented in Table 5. In this table, we choose four first terms of the base for all three different choices of base functions.

Table: Absolute error (Example 5)

t	Eta-based functions	Legendre polynomials	Trigonometric
0.2	1.4×10^{-11}	5.6×10^{-4}	2.4×10^{-5}
0.4	2.9×10^{-11}	4.7×10^{-4}	7.2×10^{-5}
0.6	4.4×10^{-11}	6.4×10^{-4}	1.1×10^{-4}
0.8	5.9×10^{-11}	1.2×10^{-4}	3.8×10^{-5}

Orthogonal polynomials and polynomial approximations

Orthogonal polynomials play the most important role in spectral methods and, therefore, it is necessary to highlight their relevant properties. This section is devoted to the study of the properties of general orthogonal polynomials. We briefly review the fundamental results on the polynomial approximations.

The family of orthogonal polynomials constitutes a basis of the Hilbert space $L_2([a,b],w(x))$, with the standard inner product given by

$$(f,g)=\int_a^b f(x)g(x)w(x)dx.$$

These orthogonal polynomials satisfy many important properties, among which we highlight the following:

- The polynomial $p_n(x)$ is of degree exactly equal to n for all $n\geq 0$.

- The zeros of $p_n(x)$ are simple and they are located in the interval (a,b).

- The following orthogonality property is satisfied:

$$\int_a^b p_n(x)p_m(x)w(x)dx = c_n\delta_{nm}, \quad m, n = 0, 1, 2, ..., N$$

or equivalently

$$\int_a^b x^m p_n(x)w(x)dx = 0, \;\; for \;\; n = 1, 2, ...; \; m < n.$$

The interval (a, b) is called the interval of orthogonality and need not be finite.
The weight function $w(x) \geq 0$ for all $x \in [a, b]$ and $w(x) > 0$ for all $x \in (a, b)$.

• A sequence of orthogonal polynomials $p_n(x)$ satisfies a 3-term recurrence relation of the form.

$$xp_n(x) = p_{n+1}(x) + a_n p_n(x) + b_n p_{n-1}(x),$$

where we set $p_{-1}(x) = 0$ and $p_0(x) = 1$ and coefficients a_n and b_n that can be written in terms of the inner product

$$a_n = \frac{(xp_n, p_n)}{(p_n, p_n)}, \;\; n \geq 0, \;\; b_n = \frac{(p_n, p_n)}{(p_{n-1}, p_{n-1})}, \;\; n \geq 1.$$

• The derivatives of orthogonal polynomials also form orthogonal polynomial sets.

• Ortoghonal polynomials satisfy a second order linear differential equation of the Sturm-Liouville type

$$p(x){y_n}''(x) + q(x){y_n}'(x) + \lambda_n y_n(x) = 0$$

where $p(x)$ is a polynomial of degree ≤ 2, $q(x)$ is a linear polynomial, both independant of n, and λ_n is independant of x. Equivalently, the weights satisfy a first-order differential equation, the Pearson equation

$$\frac{d}{dx}\left[p(x)w(x)\right] = q(x)w(x),$$

- The Rodrigues' type formula of orthogonal polynomials is defined as:

$$y_n(x) = \frac{1}{a_n w(x)} D^n \left[w(x) p^n(x) \right], \quad n = 0, 1, 2, ...$$

where $p(x)$ is a polynomial in x independent of n and a_n does not depend on x.

The Rodrigues formula provides transparent and immediate information about the interval of orthogonality, the weight function, and the range of parameters for which orthogonality holds.
From the application point of view, the most important class is the so-called classical orthogonal polynomials, that include the well-known families: Hermite, Laguerre and Jacobi.

The polynomials orthogonal with respect to the normal distribution e^{-x^2} are the Hermite polynomials, named for the French mathematician Charles Hermite (1822 - 1901).

Definition: Hermite polynomials

The Hermite polynomials can be represented explicitly by

$$H_n(x) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^r n!}{r!(n-2r)!} (2x)^{n-2r}.$$

Theorem

The Hermite polynomials are denoted $H_n(x)$ and are defined by the generating function

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x) t^n}{n!}$$

valid for all finite x and t.

Theorem

The orthogonality property of $H_n(x)$ is

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx = 2^n n!\sqrt{\pi}\delta_{nm},$$

i.e. the Hermite polynomials are orthogonal on the real line with respect to the normal distribution.

Theorem

The three-term recurrence relation for the Hermite polynomials is given by

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad n \geq 1.$$

The Rodrigues formula for Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Laguerre polynomials, named for the French mathematician Edmond Nicolas Laguerre (1834 - 1886).

Definition

The Laguerre polynomials can be represented explicitly by

$$L_n(x) = \sum_{r=0}^{n} \frac{(-1)^r n! x^r}{(n-r)!(r!)^2}.$$

Theorem

Laguerre polynomials are denoted $L_n^{\alpha}(x)$ and are defined by the generating function

$$\frac{e^{\frac{-xt}{1-t}}}{1-t} = \sum_{n=0}^{\infty} L_n(x) t^n$$

Theorem

The Laguerre polynomials are orthogonal on the positive real line with respect to the gamma distribution, the orthogonality relation for the Laguerre polynomials is contained in

$$\int_0^\infty L_n(x)L_m(x)\, e^{-x}dx = \delta_{nm},$$

Theorem

The Laguerre polynomials satisfy the three term recurrence relation given by

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x),$$

The Rodrigues formula for Laguerre polynomials

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x}).$$

Definition

The Jacobi polynomials are defined via the hypergeometric function as follows

$$P_n^{\alpha,\beta}(x) = \frac{(\alpha+1)_n}{n!}{}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}),$$

where $(a)_n$ is Pochhammer's symbol$(a)_n = a(a+1)...(a+n-1)$

Theorem

The Jacobi polynomials can be represented explicitly by

$$P_n^{\alpha,\beta}(x) = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)}\sum_{r=0}^{n}\left(\begin{array}{c} n \\ r \end{array}\right)\frac{\Gamma(\alpha+\beta+n+r+1)}{\Gamma(\alpha+r+1}\left(\frac{1-x}{2}\right)^r.$$

Definition

For real x the Jacobi polynomial can alternatively be written as

$$P_n^{\alpha,\beta}(x) = \sum_{r=0}^{n} \left(\begin{array}{c} n+\alpha \\ n-r \end{array} \right) \left(\begin{array}{c} n+\beta \\ r \end{array} \right) \left(\frac{x-1}{2} \right)^r \left(\frac{x-1}{2} \right)^{n-r}.$$

and for integer n

$$\left(\begin{array}{c} n \\ r \end{array} \right) = \begin{cases} \frac{\Gamma(n+1)}{\Gamma(n+1)\Gamma(n-r+1)} & r \geq 0, \\ 0 & r < 0. \end{cases}$$

Theorem

The Jacobi polynomials satisfy the orthogonality condition

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{\alpha,\beta}(x) P_m^{\alpha,\beta}(x) dx$$

$$= \frac{2^{\alpha+\beta+1}}{\alpha+\beta+2n+1} \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\Gamma(\alpha+\beta+n+1)} \delta_{nm}, \quad \alpha, \beta > -1.$$

Definition

The Legendre polynomials are Jacobi polynomials with $\alpha = \beta = 0$:

$$P_n(x) = P_n^{0,0}(x) = {}_2F_1(-n, n+1; 1; \frac{1-x}{2})$$

The Legendre polynomials satisfy the three term recurrence relation given by

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x),$$

Theorem

Legendre polynomials are the suitably normalized regular solution of differential equation

$$(1-x^2)y''(x) - 2xy'(x) + n(n+1)y(x) = 0$$

Theorem

The orthogonality property of $P_n(x)$ is

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\delta_{nm},$$

Definition

The Chebyshev polynomials of the first kind can be obtained from the Jacobi polynomials by taking $\alpha = \beta = \frac{-1}{2}$

$$T_n(x) = \frac{P_n^{\frac{-1}{2},\frac{-1}{2}}(x)}{P_n^{\frac{-1}{2},\frac{-1}{2}}(0)} = {}_2F_1(-n, n+1; \frac{1}{2}; \frac{1-x}{2}).$$

Theorem

The orthogonality property of $T_n(x)$ is

$$\int_{-1}^{1} (1-x^2) T_n(x) T_m(x)\, dx = \begin{cases} \frac{\pi}{2}\delta_{nm} & n \neq 0 \\ \pi\delta_{nm} & n = 0 \end{cases},$$

Theorem

The Chebyshev polynomials satisfy the three term recurrence relation given by

$$2xT_n(x) = T_{n+1}(x) + T_{n-1}(x),\ T_0(x) = 1,\ T_1(x) = x.$$

Theorem

Chebyshev polynomials is the suitably normalized regular solution of differential equation

$$(1 - x^2)y''(x) - xy'(x) + n^2y(x) = 0$$

Definition

Bessel functions, first defined by the mathematician Bernoulli and then generalized by Bessel, are canonical solutions of Bessel's differential equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0,$$

for an arbitrary complex number α , the order of the Bessel function.

Theorem

Bessel functions of the first kind of order n denote by $J_n(x)$ and can be represented explicitly by

$$J_n(x) = \sum_{r=0}^{n} \frac{(-1)^r}{\Gamma(n+r+1)(r!)} (\frac{x}{2})^{2r+n}.$$

Theorem

Bessel functions of the first kind are defined by the generating function

$$e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x)t^n.$$

Theorem

Orthogonality relation for the first kind Bessel functions is

$$\int_0^1 xJ_n(\xi_i x)J_n(\xi_i x)\,dx = \frac{1}{2}\{J_{n+1}(\xi_i)\}^2\delta_{ij},$$

if ξ_i and ξ_j are roots of equation $J_n(x)$.

The fractional derivative and integral

Definition 1. Caputo's fractional derivative of order α is defined as

$$(D^{\alpha}x)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{x^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, \qquad n-1 < \alpha \leq n,\ n \in \mathbb{N}, \tag{79}$$

where $\alpha > 0$ is the order of the derivative and n is the smallest integer greater than α.

The fractional derivative and integral

Definition 2. The Riemann-Liouville fractional integral operator of order α is defined as

$$I^{\alpha}x(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{x(s)}{(t-s)^{1-\alpha}} ds, & \alpha > 0, \\ x(t), & \alpha = 0. \end{cases} \tag{80}$$

For the Riemann-Liouville fractional integral, we have

$$I^{\alpha}x^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1+\alpha)}x^{\nu+\alpha}, \qquad \nu > -1. \tag{81}$$

The Caputo derivative and Riemann-Liouville integral satisfy the following property

$$I^{\alpha}(D^{\alpha}x(t)) = x(t) - \sum_{k=0}^{n-1} x^{(k)}(0)\frac{t^k}{k!}. \tag{82}$$

Definition 3. The left–sided and right–sided Riemann-Liouville integrals of order μ, when $0 < \mu < 1$, are defined, respectively, as

$$({}_{x_L}^{RL}I_x^{\mu} f)(x) = \frac{1}{\Gamma(\mu)} \int_{x_L}^{x} \frac{f(s)ds}{(x-s)^{1-\mu}}, \quad x > x_L,$$

and

$$({}_{x}^{RL}I_{x_R}^{\mu} f)(x) = \frac{1}{\Gamma(\mu)} \int_{x}^{x_R} \frac{f(s)ds}{(s-x)^{1-\mu}}, \quad x < x_R,$$

where Γ represents the Euler gamma function.

Definition 4. The corresponding inverse operators, the left-sided and right-sided fractional derivatives of order μ, when $0 < \mu < 1$, are defined as

$$({}^{RL}_{x_L}D^{\mu}_{x} f)(x) = \frac{d}{dx}({}^{RL}_{x_L}I^{1-\mu}_{x} f)(x) = \frac{1}{\Gamma(1-\mu)}\frac{d}{dx}\int_{x_L}^{x}\frac{f(s)ds}{(x-s)^{\mu}}, \quad x > x_L,$$

and

$$({}^{RL}_{x}D^{\mu}_{x_R} f)(x) = \frac{-d}{dx}({}^{RL}_{x}I^{1-\mu}_{x_R} f)(x) = \frac{1}{\Gamma(1-\mu)}\frac{-d}{dx}\int_{x}^{x_R}\frac{f(s)ds}{(x-s)^{\mu}},$$

Definition 3. The corresponding left- and right-sided Caputo derivatives of order μ, when $0 < \mu < 1$, are obtained as

$$({}_{x_L}^{C}D_x^{\mu} f)(x) = ({}_{x_L}^{RL}I_x^{1-\mu} \frac{df}{dx})(x) = \frac{1}{\Gamma(1-\mu)} \int_{x_L}^{x} \frac{f'(s)ds}{(x-s)^{\mu}}, \quad x > x_L,$$

and

$$({}_{x}^{C}D_{x_R}^{\mu} f)(x) = ({}_{x}^{RL}I_{x_R}^{1-\mu} \frac{-df}{dx})(x) = \frac{1}{\Gamma(1-\mu)} \int_{x}^{x_R} \frac{-f'(s)ds}{(x-s)^{\mu}}, \quad x < x_R.$$

The left- and right-sided fractional derivatives of both Riemann-Liouville and Caputo type satisfy the following properties :

$$({}^{RL}_{x_L}D^{\mu}_{x} f)(x) - ({}^{C}_{x_L}D^{\mu}_{x} f)(x) = \frac{f(x_L)}{\Gamma(1-\mu)(x-x_L)^{\mu}}.$$

and

$$({}^{RL}_{x}D^{\mu}_{x_R} f)(x) - ({}^{C}_{x}D^{\mu}_{x_R} f)(x) = \frac{f(x_R)}{\Gamma(1-\mu)(x_R-x)^{\mu}}.$$

Finally, we recall a useful property of the Riemann-Liouville fractional derivatives. Assume that $0 < \mu < 1$ and $0 < \lambda < 1$ and $f(x_L) = 0, x > x_L$, then

$$ {}_{x_L}^{RL}D_x^{\mu+\lambda} f(x) = ({}_{x_L}^{RL}D_x^{\mu})({}_{x_L}^{RL}D_x^{\lambda}) f(x) = ({}_{x_L}^{RL}D_x^{\lambda})({}_{x_L}^{RL}D_x^{\mu}) f(x). $$

Function approximation

Suppose $f(x)$ is a continuous function which can be expanded in orthogonal polynomials $\phi_j(x)$:

$$f(x) = \sum_{j=0}^{\infty} c_j \phi_j(x), \quad a \leq x \leq b.$$

where the coefficients c_j are given by

$$c_j = \frac{1}{h_j} \int_a^b w(x) f(x) \phi_j(x) dx, \;\; j = 0, 1, 2, \ldots.$$

and

$$h_j = \int_a^b w(x) \{\phi_j(x)\}^2 dx, \;\; j = 0, 1, 2, \ldots.$$

Function approximation

In practice, only the first $(N+1)$-terms orthogonal polynomials are considered. Hence, $f(x)$ can be expressed in the form

$$f(x) \simeq \sum_{j=0}^{N} c_j \phi_j(x), = C^T \phi(x), \quad a \leq x \leq b.$$

where the coefficient vector C and the vector $\phi(x)$ are given by

$$C^T = [c_0, c_1, ..., c_N],$$

$$\phi(x) = [\phi_0(x), \phi_1(x), ..., \phi_N(x)]$$

Theorem

Let $\phi(x)$ be shifted Legendre vector on $[0, 1]$ and also suppose $\alpha > 0$ then

$$D^{\nu}\phi(x) \simeq D^{(\nu)}\phi(x),$$

where $D^{(\nu)}$ is the $(N+1) \times (N+1)$ operational matrix of derivatives of order ν in the Caputo sense and is defined as follows:

$$D^{(\nu)} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ \sum\limits_{k=\lceil \nu \rceil}^{\lceil \nu \rceil} \theta_{\lceil \nu \rceil,0,k} & \sum\limits_{k=\lceil \nu \rceil}^{\lceil \nu \rceil} \theta_{\lceil \nu \rceil,1,k} & \sum\limits_{k=\lceil \nu \rceil}^{\lceil \nu \rceil} \theta_{\lceil \nu \rceil,2,k} & \dots & \sum\limits_{k=\lceil \nu \rceil}^{\lceil \nu \rceil} \theta_{\lceil \nu \rceil,N,k} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \sum\limits_{k=\lceil \nu \rceil}^{i} \theta_{i,0,k} & \sum\limits_{k=\lceil \nu \rceil}^{i} \theta_{i,1,k} & \sum\limits_{k=\lceil \nu \rceil}^{i} \theta_{i,2,k} & \dots & \sum\limits_{k=\lceil \nu \rceil}^{i} \theta_{i,N,k} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \sum\limits_{k=\lceil \nu \rceil}^{N} \theta_{N,0,k} & \sum\limits_{k=\lceil \nu \rceil}^{N} \theta_{N,1,k} & \sum\limits_{k=\lceil \nu \rceil}^{N} \theta_{N,2,k} & \dots & \sum\limits_{k=\lceil \nu \rceil}^{N} \theta_{N,N,k} \end{pmatrix}$$

where

$$\theta_{i,j,k} = (2j+1)\sum_{\ell=0}^{j} \frac{(-1)^{i+j+k+\ell}(i+k)!(\ell+j)!}{(i-k)!k!\Gamma(k-\nu+1)(j-\ell)!(\ell!)^2(k+\ell-\nu+1)}$$

Note that in $D^{(\mu)}$, the first $\lceil \nu \rceil$ rows are all zero.

Fractional neutral delay differential equation with state-dependent and time-dependent delay

We consider the fractional neutral delay differential equation with state-dependent and time-dependent delay:

$$D^\alpha X(t) = F(t, X(t), X(\tau(t) - \Delta_1(X(t))), D^\alpha X(\phi(t) - \Delta_2(X(t)))), \tag{83}$$

where $X(0) = X_0,$, $0 < \alpha \leq 1, \;\; 0 \leq t \leq t_f$, and

$$X(t) = [x_1(t), x_2(t), \cdots, x_l(t)]^T, \qquad F(t) = [f_1(t), f_2(t), \cdots, f_l(t)]^T,$$

are l-vector functions, $F(t)$ assumed to be a sufficiently smooth real valued vector function, $\Delta_1(t, X(t))$, $\Delta_2(t, X(t))$, $\tau(t)$ and $\phi(t)$ are assumed to be continuous functions for all $t \in [0, t_f]$. We assume $X(t) = 0$ for $t < 0$.

In this section, we introduce the new numerical method to solve the fractional neutral delay differential equation in Eq. (83). For starting the new numerical method discussion, first, we present the least-squares approximation using Jacobi polynomials. The class of Jacobi polynomials, $P_m^{(\gamma,\beta)}(t)$, includes all the polynomial solutions to singular Strum-Lioville problems on $(-1,1)$. These polynomials satisfy the relation

$$P_m^{(\gamma,\beta)}(t) = \sum_{k=0}^{m} \left(\begin{array}{c} m+\gamma \\ k \end{array} \right) \left(\begin{array}{c} m+\beta \\ m-k \end{array} \right) (\frac{t-1}{2})^{m-k} (\frac{t+1}{2})^{k},$$

where

$$\left(\begin{array}{c} z \\ n \end{array} \right) = \begin{cases} \frac{\Gamma(z+1)}{\Gamma(n+1)\Gamma(z-n+1)}, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

Numerical method

These polynomials are orthogonal with respect to the weight $w(t) = (1-t)^{\alpha}(1+t)^{\beta}$ on the interval $[-1, 1]$. Suppose $x(t)$ and a vector of base functions

$$G(t) = [P_0^{(\gamma,\beta)}(t),\ P_1^{(\gamma,\beta)}(t), \ldots, P_M^{(\gamma,\beta)}(t)]^T$$

are defined on $t \in (-1, 1)$. For the least squares approximation, the coefficients $c_0,\ c_1, \ldots, c_M$ of the sum

$$x_M^G(t) = c_0 P_0^{(\gamma,\beta)}(t) + c_1 P_1^{(\gamma,\beta)}(t) + \cdots + c_M P_M^{(\gamma,\beta)}(t), \tag{84}$$

must be determined in such a way that the integral

$$I_M^G = \int_{-1}^{1} (x(t) - x_M^G(t))^2 dt,$$

is minimal.

The function $x_M^G(t)$ with these coefficients is called the least squares fit of $x(t)$ with respect to vector G. The piecewise least-squares approximation is a powerful method to increase the accuracy of the approximation.

It is easy to show, by using the properties of Jacobi polynomials and Eq. (81), we could derive the fractional integral of Jacobi polynomials. For Example, for $\gamma = \beta = 0$, we have

$$I^\alpha(P_m^{(\gamma,\beta)}(t)) = 2^m \sum_{k=0}^{m} \binom{m}{k} \binom{\frac{m+k-1}{2}}{m} \frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)} t^{k+\alpha}. \qquad (85)$$

Using Eq. (84), the least squares approximation of $D^\alpha x_i(t)$ in Eq. (83) has a general form

$$D^\alpha x_i(t) = G^T(t)C_i, \tag{86}$$

where

$$G^T(t) = [P_0^{(\gamma,\beta)}(\frac{2}{t_f}(t-t_f)+1), P_1^{(\gamma,\beta)}(\frac{2}{t_f}(t-t_f)+1), ..., P_M^{(\gamma,\beta)}(\frac{2}{t_f}(t-t_f)+1)],$$

$$C_i = [c_0^i, c_1^i, ..., c_M^i]^T.$$

Using Eq. (86) we have

$$D^\alpha X(t) = \hat{G}(t)\hat{C}, \tag{87}$$

where $\hat{C}$ is a $l(M+1)\times 1$ vector given by

$$\hat{C} = [C_1, C_2, \cdots, C_l]^T,$$

and

$$\hat{G}(t) = I_l \otimes G^T(t),$$

in which I_l is the l dimensional identity matrix, $\hat{G}(t)$ is $l \times l(M+1)$ matrix as well, and $\otimes$ denotes Kronecker product. Using Eqs. (82) and (87) we get

$$X(t) = \hat{G}(t,\alpha)\hat{C} + X_0, \tag{88}$$

where $\hat{G}(t,\alpha) = I_l \otimes G^T(t,\alpha)$ and

$$G^T(t,\alpha) =$$

$$[I^\alpha P_0^{(\gamma,\beta)}(\frac{2}{t_f}(t-t_f)+1), I^\alpha P_1^{(\gamma,\beta)}(\frac{2}{t_f}(t-t_f)+1), ..., I^\alpha P_M^{(\gamma,\beta)}(\frac{2}{t_f}(t-t_f)+1)].$$

By replacing Eqs. (87) and (88) in (83), we have

$$\hat{G}(t)\hat{C} = F(t, \hat{G}(t,\alpha)\hat{C} + X_0, \hat{G}((\tau(t) - \Delta_1(\hat{G}(t,\alpha)\hat{C} + X_0)), \alpha)\hat{C} + X_0, \hat{G}(\phi(t) - \Delta_2(\hat{G}(t,\alpha)\hat{C} + X_0)\hat{C}). \tag{89}$$

We use the collocation method by requiring the residual of the problem i.e.,

$$R(t,\hat{C}) = \hat{G}(t)\hat{C} - F(t, \hat{G}(t,\alpha)\hat{C} + X_0, \hat{G}((\tau(t) - \Delta_1(\hat{G}(t,\alpha)\hat{C} + X_0)), \alpha)\hat{C}$$

$$+X_0, \hat{G}(\phi(t) - \Delta_2(\hat{G}(t,\alpha)\hat{C} + X_0)\hat{C}), \tag{90}$$

to vanish on the collocation points which leads to a system of $l(M+1)$ nonlinear equations which can be solved for the elements of $\hat{C}$ using the well Newton's iterative method. Finally, we calculate $X(t)$ given in Eq. (88). We have used three different collocation points and compare the accuracy of the method by using each of them.

Error bounds

In this section, we give some estimates for the error of the Jacobi approximation of a function $x(t)$ in terms of Sobolev norms.
The Sobolev norm of integer order $\mu \geq 0$ in the interval $(0, t_f)$, is given by

$$\| x \|_{H_w^\mu(0,t_f)} = \left(\sum_{k=0}^{\mu} \int_0^{t_f} |x^{(k)}(t)|^2 w(t) dt \right)^{\frac{1}{2}} = \left(\sum_{k=0}^{\mu} \| x^{(k)} \|_{L_w^2}^2 \right)^{\frac{1}{2}}, \tag{91}$$

where $x^{(k)}$ denotes the $k-th$ derivative of x and $H_w^\mu(0, t_f)$ is a weighted Sobolev space relative to the weight function w.

We will also assume that the problem is sufficiently smooth and that there exist Lipschitz constants, L_1, L_2, L_3, L_x, L_{x^α}, L_{Δ_1} and L_{Δ_2} for which the following inequalities hold.

$$\begin{aligned} \|f(t,x_2,y_2,z_2)-f(t,x_1,y_1,z_1)\|_{L^2_w(0,t_f)} &\leq L_1\,\|x_2-x_1\|_{L^2_w(0,t_f)} \\ &+ L_2\,\|y_2-y_1\|_{L^2_w(0,t_f)} + L_3\,\|z_2-z_1\|_{L^2_w(0,t_f)}, \end{aligned} \tag{92}$$

$$\|x(t_2)-x(t_1)\|_{L^2_w(0,t_f)} \leq L_x\,\|t_2-t_1\|_{L^2_w(0,t_f)}, \tag{93}$$

Error bounds

$$\|D^\alpha x(t_2) - D^\alpha x(t_1)\|_{L^2_w(0,t_f)} \leq L_{x^\alpha} \|t_2 - t_1\|_{L^2_w(0,t_f)}, \tag{94}$$

$$\|\Delta_1(x_2) - \Delta_1(x_1)\|_{L^2_w(0,t_f)} \leq L_{\Delta_1} \|x_2 - x_1\|_{L^2_w(0,t_f)}, \tag{95}$$

$$\|\Delta_2(x_2) - \Delta_2(x_1)\|_{L^2_w(0,t_f)} \leq L_{\Delta_2} \|x_2 - x_1\|_{L^2_w(0,t_f)}. \tag{96}$$

Sufficient conditions for the existence and uniqueness of solutions for Eq. (83) are: f is continuous with respect to $t, x(t), x(\tau(t) - \Delta_1(x(t)))$ and $D^\alpha x(\phi(t) - \Delta_2(x(t)))$, $x(t)$ is continuous, f satisfies Lipschitz conditions (Eq. (92)), and f is bounded. Also, we shall make use of the assumption that the Lipschitz constant L_3 be less than 1. To state our main results, we recall the following theorem.

Error bounds

Theorem 5: Suppose $x \in H_w^{\mu}(\mathcal{I})$ with $\mathcal{I} = (0, t_f)$ and $\mu \geq 0$, and $f_M^G(x)$ be the least squares approximation of $x(t)$, then

$$\left\| x - x_M^G \right\|_{L_w^2(\mathcal{I})} \leq cM^{-\mu} \left\| x^{(\mu)} \right\|_{L_w^2(\mathcal{I})}, \tag{97}$$

and for $1 \leq r \leq \mu$,

$$\left\| x - x_M^G \right\|_{H_w^r(\mathcal{I})} \leq cM^{2r-\frac{1}{2}-\mu} \left\| x^{(\mu)} \right\|_{L_w^2(\mathcal{I})}, \tag{98}$$

where c depends on μ.

Remark : Here, we expand $D^\alpha x$ as Jacobi polynomials and from Eqs. (86) and (97) for $M \geq \mu - 1$ we have

$$\| D^\alpha x - (D^\alpha x)_M^G \|_{L^2_w(\mathcal{I})} \leq cM^{-\mu} \| (D^\alpha x)^{(\mu)} \|_{L^2_w(\mathcal{I})}, \tag{99}$$

now by using Eq. (82) and inequality (99) we obtain the error estimate $x - x_M^G$ for $x \in H_w^\mu(\mathcal{I})$ as

$$\begin{aligned} &\|x - x_M^G\|_{L^2_w(I)} \\ &= \|I^\alpha D^\alpha x - I^\alpha (D^\alpha x)_M^G\|_{L^2_w(\mathcal{I})} = \|I^\alpha (D^\alpha x - (D^\alpha x)_M^G)\|_{L^2_w(\mathcal{I})} \\ &\qquad \leq c_1 I^\alpha \|D^\alpha x - (D^\alpha x)_M^G\|_{L^2_w(\mathcal{I})} \leq CM^{-\mu} \| (D^\alpha x)^{(\mu)} \|_{L^2_w(\mathcal{I})}, \end{aligned} \tag{100}$$

and for $1 \leq r \leq \mu$,

$$\left\| x - x_M^G \right\|_{H_w^r(\mathcal{I})} \leq C M^{2r-\frac{1}{2}-\mu} \left\| (D^\alpha x)^{(\mu)} \right\|_{L_w^2(\mathcal{I})}, \tag{101}$$

where C depends on μ and α.
Now, we establish sufficient conditions for convergence of the method that is based on our approach. In order to establish our convergence results, we assume that the solution is sufficiently smooth except perhaps for a finite number of points. We also assume that the function $f(t, x(t), y(t), z(t))$ is sufficiently smooth except for the points where $x(t), y(t)$ or $z(t)$ is not smooth.

Error bounds

Throughout this section, we shall use these inequalities

$$\begin{aligned}&\|y(\tau(t)-\Delta_1(y(t)))-x(\tau(t)-\Delta_1(x(t)))\|_{L^2_w(\mathcal{I})}\leq\\&\quad\|y(\tau(t)-\Delta_1(y(t)))-x(\tau(t)-\Delta_1(y(t)))\|_{L^2_w(\mathcal{I})}+L_xL_{\Delta_1}\|y(t)-x(t)\|_{L^2_w(\mathcal{I})},\end{aligned}\tag{102}$$

for the delay term associated with Eq. (83) where the functions x and Δ_1 are assumed to satisfy the Lipschitz conditions (Eqs. (93) and (95)). These inequalities follow by applying the triangle inequality and the Lipschitz conditions, and

$$\begin{aligned}&\|D^\alpha y(\phi(t)-\Delta_2(y(t)))-D^\alpha x(\phi(t)-\Delta_2(x(t)))\|_{L^2_w(\mathcal{I})}\leq\\&\quad\|D^\alpha y(\phi(t)-\Delta_2(y(t)))-D^\alpha x(\phi(t)-\Delta_2(y(t)))\|_{L^2_w(\mathcal{I})}+L_{x^\alpha}L_{\Delta_2}\|y(t)-x(t)\|_{L}\end{aligned}\tag{103}$$

for the derivative delay term associated with Eq. (83) where the functions $D^\alpha x$ and Δ_2 are assumed to satisfy the Lipschitz conditions (Eqs. (94) and (96)).

Theorem 6: Let $x(t)$ be the exact solution of Eq. (83) and $u(t)$ be the best approximation of $x(t)$ based on the Jacobi polynomilas. If $x \in H_w^{\mu}(\mathcal{I})$ then for $\mu \geq 0$, we can write

$$\|x(t) - u(t)\|_{L_w^2(\mathcal{I})} \leq \zeta \tfrac{L}{1-L_3} M^{-\mu} \left\| x^{(\alpha+\mu)} \right\|_{L_w^2(\mathcal{I})}, \tag{104}$$

for $1 \leq r \leq \mu$ we have,

$$\|x(t) - u(t)\|_{H_w^r(\mathcal{I})} \leq \zeta \frac{L}{1 - L_3} M^{2r - \frac{1}{2} - \mu} \left\| x^{(\alpha+\mu)} \right\|_{L_w^2(\mathcal{I})}, \tag{105}$$

provided that M is sufficiently large, f, x, $D^{\alpha}x$, Δ_1 and Δ_2 satisfy the Lipschitz conditions (Eqs. (92)-(96)) with $L_3 < 1$, and ζ is a constant independent of N.

Proof. Subtracting (89) from (83), integrating, and taking a norm of both sides, we obtain

$$
\begin{aligned}
&\|x(t)-u(t)\|_{L^2_w(\mathcal{I})} \leq \\
&I^{\alpha}[\|f(s,x(s),x(\tau(s)-\Delta_1(x(s))),D^{\alpha}x(\phi(s)-\Delta_2(x(s))))- \\
&f(s,u(s),u(\tau(s)-\Delta_1(u(s))),D^{\alpha}u(\phi(s)-\Delta_2(u(s))))\|_{L^2_w(\mathcal{I})}\Big] \leq \\
&I^{\alpha}\Big[L_1\left\|x(s)-u(s)\right\|_{L^2_w(\mathcal{I})}+L_2\left\|x(\tau(s)-\Delta_1(x(s)))-u(\tau(s)-\Delta_1(u(s)))\right\|_{L^2_w}+ \\
&L_3\left\|D^{\alpha}x(\phi(s)-\Delta_2(x(s)))-D^{\alpha}u(\phi(s)-\Delta_2(u(s)))\right\|_{L^2_w(\mathcal{I})}\Big],
\end{aligned}
\tag{106}
$$

where $D^{\alpha}u(t)=\hat{G}(t)\hat{C}$ and $u(t)=\hat{G}(t,\alpha)\hat{C}+X_0$.

Error bounds

Let Let $\Psi(t) = \|x(t) - u(t)\|_{L^2_w(\mathcal{I})}$, $\Upsilon(t) = \max_{0 \le s \le t} \Psi(s)$, and $\chi(t) = \max_{0 \le s \le t} \|D^\alpha x(s) - D^\alpha u(s)\|_{L^2_w(\mathcal{I})}$.
Substituting Ψ, Υ and χ into the above inequality and using (102)-(103) we have

$$\begin{aligned}
&\Psi(t) \le \\
&I^\alpha[L_1\Upsilon(s) + L_2 \|x(\tau(s) - \Delta_1(u(s))) - u(\tau(s) - \Delta_1(u(s)))\|_2 + L_2 L_x L_{\Delta_1}\Upsilon(s) \\
&L_3 \|D^\alpha x(\phi(s) - \Delta_2(u(s))) - D^\alpha u(\phi(s) - \Delta_2(u(s)))\|_2 + L_3 L_{x^\alpha} L_{\Delta_2}\Upsilon(s)] \le \\
&I^\alpha[(L_1 + L_2 + L_2 L_x L_{\Delta_1} + L_3 L_{x^\alpha} L_{\Delta_2})\Upsilon(s) + L_3\chi(s)].
\end{aligned} \tag{107}$$

Error bounds

Subtracting (89) from (83) and using (102)-(103) yields

$$
\begin{aligned}
&\|D^{\alpha}x(t) - D^{\alpha}u(t)\|_{L^2_w(\mathcal{I})} = \\
&\|f(t, x(t), x(\tau(t) - \Delta_1(x(t))), D^{\alpha}x(\phi(t) - \Delta_2(x(t)))) \\
&-f(t, u(t), u(\tau(t) - \Delta_1(u(t))), D^{\alpha}u(\phi(t) - \Delta_2(u(t))))\|_{L^2_w(\mathcal{I})} \\
&\leq L_1\Upsilon(t) + L_2\left\|x(\tau(t) - \Delta_1(u(t))) - u(\tau(t) - \Delta_1(u(t)))\right\|_{L^2_w(\mathcal{I})} \\
&+L_2L_xL_{\Delta_1}\Upsilon(t) + L_3\left\|D^{\alpha}x(\phi(t) - \Delta_2(u(t))) - D^{\alpha}u(\phi(t) - \Delta_2(u(t)))\right\|_{L^2_w(\mathcal{I})} \\
&+L_3L_{X^{\alpha}}L_{\Delta_2}\Upsilon(t) \leq (L_1 + L_2 + L_2L_xL_{\Delta_1} + L_3L_{x^{\alpha}}L_{\Delta_2})\,\Upsilon(t) + L_3\chi(t).
\end{aligned}
\tag{108}
$$

Error bounds

Then, from (108), we find

$$\chi(t) \leq \left(L_1 + L_2 + L_2 L_x L_{\Delta_1} + L_3 L_{x^\alpha} L_{\Delta_2}\right) \Upsilon(t) + L_3 \chi(t). \tag{109}$$

As $0 < L_3 < 1$, inequality (109) becomes

$$\chi(t) \leq \left(\frac{L_1 + L_2 + L_2 L_x L_{\Delta_1} + L_3 L_{x^\alpha} L_{\Delta_2}}{1 - L_3}\right) \Upsilon(t). \tag{110}$$

Error bounds

Letting $L = L_1 + L_2 + L_2 L_x L_{\Delta_1} + L_3 L_{x^\alpha} L_{\Delta_2}$, substituting (110) into (107), using (81) and (99) leads to

$$\begin{aligned}\Psi(t) = \|x(t) - u(t)\|_{L^2_w(\mathcal{I})} &\leq I^\alpha\left(L\Upsilon(s) + \left(\tfrac{L_3 L}{1-L_3}\right)\Upsilon(s)\right)\\ &= \tfrac{L}{1-L_3} I^\alpha \Upsilon(s) \leq C\tfrac{t_f^\alpha}{\alpha!}\tfrac{L}{1-L_3} M^{-\mu}\left\|X^{(\alpha+\mu)}\right\|_{L^2_w(\mathcal{I})},\end{aligned} \tag{111}$$

for $1 \leq r \leq \mu$ we have,

$$\|x(t) - u(t)\|_{H^r_w(\mathcal{I})} \leq C\frac{t_f^\alpha}{\alpha!}\frac{L}{1-L_3} M^{2r-\frac{1}{2}-\mu}\left\|x^{(\alpha+\mu)}\right\|_{L^2_w(\mathcal{I})}. \tag{112}$$

In this section, to demonstrate the applicability and accuracy of the present method we consider Eq. (83).To vanish the residual of the problem in Eq. (90), we consider three different collocation points: Equidistant points, Zero of the Legendre polynomials, and Extreme points of the Chebyshev polynomial. Using these collocation points, first, we calculate the error of the present method for $\alpha = 1$. Then we use the collocation point that has fewer errors for $\alpha = 1$ to solve the problem for $\alpha \neq 1$. We use the least-squares approximation and evaluate the absolute maximum error as $|x_i(t) - {x_i}_M^G(t)|$. Also, we use the piecewise least-squares approximation in some cases to increase the accuracy.

Example: Consider a 3–dimensional time-dependent delay system

$$
\begin{aligned}
D^\alpha x_1(t) &= D^\alpha x_2(\sqrt{\sin(t)}) + x_3(\sqrt{\sin(t)}) + 2tD^\alpha x_3(t^2),\\
D^\alpha x_2(t) &= -2x_1(\sqrt{t}) + x_2(e^{-2t}) + x_3(t) - D^\alpha x_3(e^{-2t}),\\
D^\alpha x_3(t) &= x_2(t) + tD^\alpha x_2(t-\sin(t)) + tx_3(t-\sin(t)),
\end{aligned}
$$

that $x_i(t) = 0$, for $i = 1,\ 3$ and $x_i(t) = 0$, for $i = 2$ and has the exact solution , $x_1(t) = sin(t^2),\ \ x_2(t) = cos(t),\ \ x_3(t) = sin(t)$. Table 1 shows the absolute error for three different choices of collocation points with $M = 7$, $\alpha = 1$. We use the extreme points of the Chebyshev polynomial as the collocation point to solve the example for $\alpha \neq 1$. In Table 2, we used the piecewise least-squares approximation with four subintervals and $M = 7$.

Accuracy of the numerical method

	Equidistant points	Zero of the Legendre	Extreme points of the Chebyshev
$x_1(t)$	7.0×10^{-6}	3.5×10^{-7}	5.0×10^{-7}
$x_2(t)$	3.5×10^{-6}	4.0×10^{-8}	1.0×10^{-7}
$x_3(t)$	6.0×10^{-6}	2.5×10^{-8}	6.0×10^{-8}

Table: Absolute error with the different choices of collocation points

Accuracy of the numerical method

Method	$x_1(t)$	$x_2(t)$	$x_3(t)$
Multi–quadric approximation	2×10^{-7}	2×10^{-7}	4×10^{-7}
Present method with $M = 7$	4×10^{-12}	7×10^{-13}	6×10^{-13}

Table: Absolute error for Case 1 by using the present method and Multi–quadric approximation.

Thank you