On Hessians and the Lefschetz properties

Lecture 1: Hypersurfaces with vanishing Hessian

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26, June - ICTP, Trieste, Italy

Introduction

The aim of this lecture is to recall classical results and constructions concerning hypersurfaces with vanishing Hessian (see [GN, Pe, Pt1, Pt2, Pt3]). Some of this classical work was revisited in [CRS, GR, Lo, GRu, Ru, Wa1, BW]. A modern reference that contains the omitted details of this lecture is [Ru, Chapter 7].

Gauss in his classical paper on curvature of surfaces calculates the (Gaussian) curvature of a implicit surface (see [Ga]). The formulae contains a Hessian determinant. In this paper he also made a discussion about local properties of surfaces of zero curvature.

The algebraic counterpart of zero curvature is the concept of developable surface. B. Segre proved that a projective complex surface $X = V(f) \subset \mathbb{P}^3$ is developable if and only if it is a cone or the tangent surface of a curve (see [Se])

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The Hessian

Definition

Let $X = V(f) \subset \mathbb{P}^N$ be a reduced hypersurface. The Hessian matrix of f is

$$\mathsf{Hess}_f = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{0<i,j<N}$$

We also call it the Hessian matrix of X and write Hess_X since we will be interested in properties of this matrix which are well defined modulo the multiplication of f by a non zero constant. The determinant of the matrix Hess_X will be denoted by hess_X and called the *hessian of X*.

A wrong claim by Hesse

Since cones have vanishing Hessian and Tangent surfaces no, one can suspect that it inspires Hesse's claim:

Claim (Hesse)

Let $f \in \mathbb{C}[x_1, \ldots, x_N]$ be an irreducible homogeneous polynomial. Then $\text{hess}_f = 0$ if and only if up to a projective transformation f does not depend on all the variables (see [He1, He2]).

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Projective cones

For an arbitrary set $S \subseteq \mathbb{P}^N$ we shall indicate by $< S >$ its linear span in $\mathbb{P}^N$ and $S \subseteq \mathbb{P}^N$ is said to be *degenerated* if $< S > \subsetneq \mathbb{P}^N$. By abuse of notation we use $< p, q >$ to denote the line through two distinct points $p, q \in \mathbb{P}^N$.

Definition

Let $X \subset \mathbb{P}^N$ be a projective variety. The vertex of X is

$$\mathrm{Vert}(X) = \{p \in X \mid < p, q > \subset X, \ \forall q \in X\}.$$

A projective variety $X \subset \mathbb{P}^N$ is a cone if $\mathrm{Vert}(X) \neq \emptyset$.

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Projective cones

Proposition

Let $X = V(f) \subset \mathbb{P}^N$ be a hypersurface of degree d. Then the following conditions are equivalent:

(i) *X is a cone;*

(ii) *There exists a point $p \in X$ of multiplicity d;*

(iii) *The partial derivatives $\frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N}$ of f are linearly dependent;*

(iv) *Up to a projective transformation, f depends on at most N variables.*

(v) *The dual variety of X, $X^* \subset (\mathbb{P}^N)^*$, is degenerated, that is contained in a hyperplane of $(\mathbb{P}^N)^*$.*

Cones form a trivial class of hypersurfaces with vanishing hessian. Indeed if X is a cone up to a linear change of coordinates the form f does not depend on all the variables. Therefore in this case the Hessian matrix has at least a null row (and a null column), yielding $\mathrm{hess}_X \equiv 0$.

The case $d = 2$, the claim is trivial from basic linear algebra. The converse is not true in general if $d \geq 3$. In fact, there are counter examples for any $d \geq 3$ and $N \geq 4$ as we will see in the sequel. We now define the polar map of a hypersurface in order to begin to clarify the deep geometrical consequences of the condition $\mathrm{hess}_X \equiv 0$.

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The polar map

Definition

The *polar map* (or *gradient map*) of a hypersurface $X = V(f) \subset \mathbb{P}^N$ is the rational map $\varphi_X : \mathbb{P}^N \dashrightarrow (\mathbb{P}^N)^*$ given by the derivatives of f:

$$\varphi_X(p) = (\frac{\partial f}{\partial x_0}(p) : \frac{\partial f}{\partial x_1}(p) : \ldots : \frac{\partial f}{\partial x_N}(p)).$$

Let $Z = \overline{\varphi_X(\mathbb{P}^N)} \subseteq (\mathbb{P}^N)^*$ be the image of the polar map. The base locus scheme of the polar map is the singular scheme of X which will be denoted by

$$\mathrm{Sing}(X) := V(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_N}) \subset \mathbb{P}^N.$$

We define $Y := \mathrm{Sing}(X)_{\mathrm{red}}$.

Un esempio semplicissimo

The easiest counterexample to Hesse's claim is $f = xu^2 + yuv + zv^2 \in \mathbb{K}[x, y, z, u, v]$ and it was explicitly posed by Perazzo in [Pe], who called it *un esempio semplicissimo*.

The key point to construct counterexamples is to figure out that the vanishing of the Hessian is equivalent to the algebraic dependence among the partial derivatives (see *loc. cit.*). On the other hand, to be a cone is equivalent to the linear dependence among the partial derivatives. This result is sometimes referred as Gordan-Noether's criterion since it was implicitly used in [GN].

Gordan-Noether criterion

Proposition (Gordan-Noether)

Let $f \in \mathbb{K}[x_0, \dots, x_N]$ *be a reduced polynomial and consider* $X = V(f) \subset \mathbb{P}^N$. *Then*

(i) X *is a cone if and only if* $f_{x_0}, \dots, f_{x_N}$ *are linearly dependent (equivalently the polar image is degenerated);*

(ii) $\mathrm{hess}_f = 0$ *if and only if* $f_{x_0}, \dots, f_{x_N}$ *are algebraically dependent (equivalently the polar map is not dominant).*

Gordan-Noether Theorem

Theorem (Gordan-Noether)

Let $X = V(f) \subset \mathbb{P}^N$, $N \leq 3$, be a hypersurface such that $\mathrm{hess}_f = 0$. Then X is a cone.

In [GN] the authors produced a series of counter-examples to Hesse's claim for each $N \geq 4$ and for each degree $d \geq 3$. The key point of the construction was to figure out that the vanishing of the Hessian is equivalent to the algebraic dependence among the partial derivatives (see *loc. cit.*).

Counter-examples to Hesse claim

Proposition (Gordan-Noether)

For each $N \geq 4$ and $d \geq 3$ there exist irreducible hypersurfaces $X = V(f) \subset \mathbb{P}^N$, of degree $\deg(f) = d$, not cones, such that $\mathrm{hess}_f = 0$.

We recall the classical constructions of Gordan and Noether ([GN]), Permutti ([Pt1, Pt2, Pt3]) and Perazzo ([Pe]) from an algebraic point of view.

Definition

Let $X = V(f) \in \mathbb{P}^N$, $N \geq 4$ be an irreducible hypersurface not a cone. We say that X is a Perazzo hypersurface of degree d if $N = n + m$, $n, m \geq 2$ and $f \in \mathbb{K}[x_0, \dots, x_n, u_1, \dots, u_m]$ is a reduced polynomial of the form

$$f = x_0 g_0 + \ldots + x_n g_n + h$$

where $g_i \in \mathbb{K}[u_1, \dots, u_m]_{d-1}$ for $i = 0, \dots, n$ are algebraically dependent but linearly independent and $h \in \mathbb{K}[u_1, \dots, u_m]_d$. The polynomial f is called Perazzo polynomial.

Perazzo hypersurfaces

Proposition (Gordan-Noether, Perazzo)

Perazzo hypersurfaces are not cones and have vanishing Hessian.

Remark

Notice that if $n+1 > m$, then g_i for $i = 0, \dots, n$ are algebraically dependent automatically. Perazzo original hypersurfaces are of degree 3 and he constructed a series of cubic hypersurfaces in $\mathbb{P}^N$ for arbitrary $N \geq 4$ with vanishing Hessian and not cones. These hypersurfaces are, modulo projective transformations, all cubic hypersurfaces with vanishing Hessian and not cones in $\mathbb{P}^N$ for $N = 4, 5, 6$, (see [Pe, GRu]).

Definition

Let $R = \mathbb{K}[x_0, \dots, x_n, u_1, \dots, u_m]$. Let $Q = \sum_{i=0}^{n} x_0 g_i \in R$ be a Perazzo polynomial. Let $\mu = \lfloor \frac{d}{e} \rfloor$. Let $P_j \in \mathbb{K}[u_1, \dots, u_m]_{d-je}$ for $j = 0, \dots, \mu$. We say that

$$f = \sum_{j=0}^{\mu} Q^j P_j$$

is a Permutti polynomial of type (m, n, e). A hypersurface $X = V(f) \subset \mathbb{P}^N$, not a cone is called a Permutti hypersurface if f is a reduced Permutti polynomial.

Permutti hypersurfaces

Proposition (Permutti)

Permutti hypersurfaces are not cones and have vanishing Hessian.

Gordan-Noether classification

The main result of Gordan-Noether in [GN] is the following one. A geometric proof in modern terms can be found in [GR, Ru].

Theorem (Gordan-Noether, Permutti)

Let $X = V(f) \subset \mathbb{P}^4$ be a reduced hypersurface, not a cone, having vanishing Hessian. Then f is a Permutti polynomial of type $(2, 2, e)$.

Let us recall a fundamental result that follows from an identity proved by Gordan and Noether (see [GN] and [Lo, 2.7]).

Corollary

Let $X = V(f) \subset \mathbb{P}^N$ be a hypersurface with vanishing hessian and let notation be as above. Then

(i) *for every $p \in \mathbb{P}^n \setminus \mathrm{Sing}(X)$ such that $\Phi_X(p) \in Z_{\mathrm{reg}}$ we have $< p, (T_{\Phi_X(p)}Z)^* > \subseteq \Phi_X^{-1}(\Phi_X(p))$;*

(ii) *for $p \in \mathbb{P}^N$ general, the closure of the irreducible component of $\Phi_X^{-1}(\Phi_X(p))$ passing through p is $< p, (T_{\Phi_X(p)}Z)^* >$. In particular for $p \in \mathbb{P}^N$ general $\overline{\Phi_X^{-1}(\Phi_X(p))}$ is a union of linear spaces of dimension equal to $\mathrm{codim}(Z)$ passing through $(T_{\Phi_X(p)}Z)^*$.*

(iii)

$$Z^* \subseteq \mathrm{Sing}(X). \tag{1}$$

Open problem: Classification

Classify irreducible hypersurfaces of degree $d \geq 3, X = V(f) \subset \mathbb{P}^n$ with $n \geq 5$ non cone and with $\text{hess}_f = 0$. Same classification problem for degree $d = 3$ and $n \geq 7$.

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Gordan-Noether theory

What we call Gordan-Noether theory is the following results.

1. Polar map and polar image.
2. Algebro-geometric description of $\mathrm{hess}_f = 0$.
3. Affirmative answer to Hesse claim for $n \leq 3$.
4. Counter-examples for any $d \geq 3$ and $n \geq 4$.
5. Gordan-Noether identity.
6. Classification for $n = 4$.

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