On Hessians and the Lefschetz properties

Lecture 2: Artinian Gorenstein algebras and the Lefschetz properties

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Introduction

In this lecture we recall basic results and definitions about standard graded Artinian Gorenstein algebras and Lefschetz properties. The main references are [Ru, MW, HMMNWW].

In several categories the cohomology functor gives rise to a Artinian algebra (that we will assume to be commutative) satisfying Poincaré duality, these algebras can be characterized as Artinian Gorenstein (AG) algebras.

Lefschetz properties are algebraic abstractions introduced by Stanley and inspired by the Hard Lefschetz theorem on the cohomology of smooth complex projective varieties. Nowadays there similar results in a great number of categories. For instance, real orientable manifolds, projective varieties, Kahler manifolds, convex polytopes, Coxeter groups and tropical varieties are examples of categories for which the ring of cohomology are Artinian Gorenstein $\mathbb{K}$-algebra.

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Artinian Gorenstein algebras

An Artinian ring is a dimension zero Noetherian ring. Let $\mathbb{K}$ be a field, a standard graded Artinian $\mathbb{K}$-algebra has a finite decomposition:

$$A = \bigoplus_{i=0}^{d} A_i.$$

In such decomposition, A_i is a finite dimensional $\mathbb{K}$-algebra and A is generated by A_1 as algebra.

Denote $h_i = \dim A_i$ and consider $h_d \neq 0$, the Hilbert function of A is finite and will be written as a vector

$$\mathrm{Hilb}(A) = (1, h_1, \ldots, h_d).$$

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If A is a standard graded Artinian $\mathbb{K}$ algebra, there is a presentation $A = \mathbb{K}[x_1, \ldots, x_n]/I$ such that $I_1 = 0$. By abuse of notation we say that I is Artinian and n is the codimension of A. The irrelevant ideal of A is $\mathfrak{m} = \sum_{i=1}^{d} A_i$ and socle of A is the ideal $(0 : \mathfrak{m}) \subset A$. We say that A is a level algebra if $(0 : \mathfrak{m}) = A_d$. Moreover we say that a level algebra is Gorenstein if $\dim_{\mathbb{K}} A_d = 1$. For a level algebra of this type, d is called socle degree of A.

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Example

Let $A = \mathbb{K}[X, Y, Z]/(X^2, Y^2, Z^2)$, we know that A is an Artinian algebra which we will consider standard graded. We have the following decomposition:

$$A = A_0 \oplus A_1 \oplus A_2 \oplus A_3.$$

It is easy to see that $A_0 =< 1 >$, $A_1 =< X, Y, Z >$, $A_2 =< XY, XZ, YZ >$ and $A_3 =< XYZ >$. The Hilbert vector of A is

$$\text{Hilb}(A) = (1, 3, 3, 1).$$

The socle ideal of A is $\mathrm{Soc}(A) = (XYZ)$, therefore, A is an Artinian Gorentein algebra.

Hilbert vectors

It is an important problem to understand the possible Hilbert vector for standard graded AG algebras. Let A be a standard graded Artinian Gorenstein algebra of codimension n and socle degree d. The family of Hilbert vectors of AG algebras of this type has a natural structure of poset. The maximal Hilbert vectors correspond to compressed algebras described by Iarrobino and they have the following formulae, $h_k = \binom{n-1+d}{d}$ for $k \leq d/2$.

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When the codimension of the algebra is less than or equal to 3, all possible Hilbert vectors have been characterized in [St]; in particular, they are unimodal, i.e. they never strictly increase after a strict decrease. While it is known that non unimodal Gorenstein h-vectors exist in every codimension greater than or equal to 5 (see [BI, Bo, BL]), it is open whether non unimodal Gorenstein h-vectors of codimension 4 exist.

Minimal Hilbert functions

The first example of a non unimodal Gorenstein h-vector was given by Stanley (see [St, Example 4.3]). He showed that the h-vector $(1, 13, 12, 13, 1)$ is indeed a Gorenstein h-vector. In [1] the authors showed that Stanley's example is optimal, i.e. it is minimal and the $h-$vector $(1, 12, 11, 12, 1)$, it is not Gorenstein.

Open problem: Minimal Hilbert vectors

It is an open problem to understand the structure of poset of the Hilbert vectors, in particular describe the minimal ones fro fixed d and n.

Poincaré duality algebras

Definition

Let A be a standard graded Artinian $\mathbb{K}$-algebra, we say that A is a Poincaré duality algebra if $\dim A_d = 1$ and for all $i = 0, \ldots, d$ the pairing given by multiplication:

$$A_i \times A_{d-i} \to A_d$$

is a perfect pairing.

The following is a very known characterization of Artinian Gorenstein algebras.

Proposition

Let $A = \bigoplus_{i=0}^{d} A_i$ be a standard graded Artinian $\mathbb{K}$ algebra. A is Gorenstein if and only if it is a Poincaré duality algebra.

Complete intersections

We say that a standard graded Artinian algebra is a complete intersection if $A = \mathbb{K}[x_1, \dots, x_n]/I$ where I is generated by a regular sequence $I = (f_1, \dots, f_n)$. It is well known that such complete intersections are Gorenstein, if we denote $\deg(f_i) = d_i$, then the socle degree of A is $d = d_1 + \cdots + d_n - n$.

Every monomial complete intersection is of the form:

$$A = \mathbb{K}[X_1, \dots, X_n]/(X_1^{d_1}, \dots, X_n^{d_n}).$$

The codimension of A is n if $d_i \geq 2$ for all $i = 1, \dots, n$. The socle of A is the ideal $(X_1^{d_1-1} \dots X_n^{d_n-1})$.

Macaulay-Matlis duality

Let $\mathbb{K}$ be a field of zero characteristic. Let $R = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring, and $Q = \mathbb{K}[X_1, \dots, X_n]$ the associated ring of differential operators, that is $X_j = \partial_j$. As usual, we consider R as a Q module via the differential action $X_i(x_j) = \delta_{ij}$. Macaulay-Matlis duality gives us a one-to-one correspondence between Q-submodules of R and ideals of Q.

For any graded Q-submodule M of R, we define the homogeneous ideal

$$\mathrm{Ann}(M) = \{\alpha \in Q | \alpha(f) = 0 \ \forall \ f \in M\} \subset Q.$$

This ideal is called the annihilator of M. For every homogeneous ideal $I \subset Q$ we define the graded Q-submodule of R,

$$I^{-1} = \{f \in R | \alpha(f) = 0 \ \forall \ \alpha \in I\}.$$

This is called the inverse system of I.

We get a natural one-to-one correspondence:

$$\begin{array}{ccc} \{\text{homogeneous ideals of } Q\} & \leftrightarrow & \{\text{graded } Q-\text{submodules of } R\} \\ \mathrm{Ann}(M) & \leftarrow & M \\ I & \rightarrow & I^{-1} \end{array}$$

We recall a very useful characterization of level algebras and Gorenstein algebras, that comes from Macaulay-Matlis duality.

Theorem

Macaulay-Matlis Let $A = \bigoplus_{i=0}^{d-1} A_i$ be an Artinian graded $\mathbb{K}$-algebra. A is a level algebra if and only if there are $g_1, \ldots, g_m \in R_d$ such that $A \simeq Q/\operatorname{Ann}(g_1, \ldots, g_m)$. A is Gorenstein if and only if $A \simeq Q/\operatorname{Ann}(f)$ for some $f \in R_d$.

Example

Take $f = x_1^d + ... + x_n^d$, the Fermat type polynomial of degree d. In this case the ideal $\text{Ann}(f)$ is generated by X_iX_j for $1 \leq i < j \leq n$ and by $X_1^d - X_j^d$ for $j = 1, 2, ..., n$. The graded part of degree k of A is $A_k =< X_1^k, X_1^k, \dots, X_n^k >$ for $k = 1, \dots, d-1$, and $A_j =< x_1^j >$ for $j = 1$ and $j = d$. This determines the Hilbert vector

$$Hilb(A(f_F)) = (1, n+1, \dots, n+1, 1).$$

Let A be monomial complete intersection and consider a presentation over Q:

$$A = \mathbb{K}[X_1, \ldots, X_n]/(X_1^{d_1}, \ldots, X_n^{d_n}).$$

The dual Macaulay generator is $f = x_1^{d_1-1} \ldots x_n^{d_n-1}$.

Open problem: Macaulay dual generator

Given an Artinian complete intersection together with a presentation over Q, that is $A = \mathbb{K}[X_1, \ldots, X_n]/(f_1, \ldots, f_n)$. How do we find the Macaulay dual generator?
A very special case is the following one. Let $X = V(f) \subset \mathbb{P}^n$ be a smooth projective hypersurface and ley $J = (f_1, \ldots, f_n)$ be its Jacobian ideal. Since X is smooth, J is a completer intersection. How do we find the Macaulay dual generator?
The second problem was solved for cubics with $n \leq 5$.

The Lefschetz properties

Definition

Let $A = \oplus_{i=0}^{d} A_i$ be an Artinian graded $\mathbb{K}-$algebra with $A_d \neq 0$.

1. The algebra A is said to have the Weak Lefschetz property, briefly WLP, if there exists an element $L \in A_1$ such that the multiplication map $\bullet L : A_i \to A_{i+1}$ is of maximal rank for $0 \leq i \leq d-1$.
2. The algebra A is said to have the Strong Lefschetz property, briefly SLP, if there exists an element $L \in A_1$ such that the multiplication map $L^k : A_i \to A_{i+k}$ is of maximal rank for $0 \leq i \leq d$ and $0 \leq k \leq d-i$.
3. We say that A has the Strong Lefschetz property in the narrow sense, if there is $L \in A_1$ such that the K-linear map $\bullet L^{d-2k} : A_k \to A_{d-k}$ is an isomorphism for all k.

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Remark

In the case of standard graded Artinian Gorenstein algebra the two conditions SLP and SLP in the narrow sense are equivalent. In this case, SLP implies WLP which implies the unimodality of the Hilbert vector. It is easy to see that for a standard graded Artinian Gorenstein algebra, the injectivity of $\bullet L : A_i \to A_{i+1}$ for a certain $L \in A_1$ implies the injectivity of $\bullet L : A_j \to A_{j+1}$ for all $j < i$. Therefore, WLP can be detected in the midle.

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Every codimension two standard graded Artinian algebra satisfy SLP (see [HMNW]). In codimension $n \geq 3$ there are examples of Artinian algebras failing WLP. In codimension $n \geq 4$ there are examples of AG algebras failing WLP. Complete intersections in codimension 3 have the WLP (see [HMNW]).

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Open problem: Lefschetz properties AG algebras of codimension 3

Is it true or false that every codimension 3 AG algebra have the SLP/WLP ?

Example

This example is due to R. Stanley and J. Watanabe. It is considered to be the starting point of the research area of Lefschetz properties for graded algebras. Nowadays there are lots of different proofs for it. Let $\mathbb{K}$ be a algebrically closed field of $char(\mathbb{K}) = 0$ and consider

$$A = \frac{\mathbb{K}[X_0, \ldots, X_n]}{(X_0^{a_0}, \ldots, X_n^{a_n})} = \frac{\mathbb{K}[X_0]}{(X_0^{a_0})} \otimes \ldots \otimes \frac{\mathbb{K}[X_n]}{(X_n^{a_n})},$$

with $a_i > 0$ for all $i = 0, \ldots, n$.

Example

It is a monomial complete intersection. Since the cohomology of the complex projective space is $H^*(\mathbb{P}^m, \mathbb{K}) = \mathbb{K}[x]/(x^{m+1})$, and the Segre product commutes with the tensor product by Künneth Theorem for cohomology, we have:

$$H^*(\mathbb{P}^{a_0-1} \times \ldots \times \mathbb{P}^{a_n-1}, \mathbb{K}) = \frac{\mathbb{K}[X_0, \ldots, X_n]}{(X_0^{a_0}, \ldots, X_n^{a_n})}.$$

By the Hard Lefschetz Theorem over $\mathbb{K} = \mathbb{C}$, we know that A has the SLP.

The standard graded Artinian Gorenstein algebra $A(f)$ associated to a form f is a natural model for the cohomology algebras of spaces in several categories. For smooth projective varieties, the Hard Lefschetz theorem inspired what is now called Lefschetz properties for the algebra $A(f)$.

Open problem: Lefschetz properties for complete intersections

Is it true that all complete intersections have the SLP/WLP ?

A special case is when the ideal of the complete intersection is the Jacobian ideal of a smooth hypersurface $X = V(f)$ of degree d. In [DGI] we proved that SLP holds for quartic curves and cubic surfaces. In [BFP], the authors proved SLP in the first open case, cubic 3-fold.

Open problem: Lefschetz properties for Milnor algebras

Is it true that all complete intersections whose ideal is the Jacobian ideal of smooth hypersurface have the SLP/WLP ?

Minimal Hilbert vectors (bis)

Definition

Let n and i be positive integers. The $i-$binomial expansion of n, denoted by $n_{(i)}$, is

$$n = n_{(i)} = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j} \qquad (1)$$

where $n_i > n_{i-1} > \ldots > n_j \geq j \geq 1$.

Classical bounds

Theorem

Let $A = R/I$ be a standard graded $\mathbb{K}$-algebra, and $L \in A$ a general linear form (according to the Zariski topology). Denote by h_d the degree d entry of the Hilbert function of A and by h'_d the degree d entry of the Hilbert function of $A/(L)$. Then:

(Macaulay)

$$h_{d+1} \leq ((h_d)_{(d)})^{+1}_{+1}.$$

(Green)

$$h'_d \leq ((h_d)_{(d)})^{-1}_{0}.$$

Full Perazzo polynomials

Now we fix $m \geq 2$ and we consider the m variables $u_1, \dots, u_m$. For a multi-index $\alpha = (e_1, \dots, e_m)$ with $e_1 + \cdots + e_m = d - 1$, let

$$M_\alpha = u_1^{e_1} \cdots u_m^{e_m} \in Q_{d-1}$$

with be a $\mathbb{K}$-linear basis for Q_{d-1} where $\dim Q_{d-1} = \binom{m+d-2}{d-1}$.

Definition

A bihomogeneous polynomial $f \in \mathbb{K}[x_1, \ldots, x_{\tau_m}, u_1, \ldots, u_m]_{(1,d-1)}$ of degree d of type:

$$f = \sum_{j=1}^{\tau_m} x_j M_j$$

is called **Full Perazzo polynomial** of type m. The associated algebra is a **Full Perazzo algebra** of socle degree d and codimension $m + \tau_m$.

A conjecture

Conjecture

Let H be the Hilbert vector of a Full Perazzo algebra of type m and socle degree d and let $r = m + \tau_m$. Then H is minimal in $\mathrm{AG}(r, d)$.

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