On Hessians and the Lefschetz properties

Lecture 3: On higher Hessians and the Lefschetz properties

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Introduction

We recall the construction of higher Hessians introduced by Maeno-Watabe in [MW] and generalized in [GZ2]. These Hessians have been used to produce AG algebras failing WLP and SLP in [Go]. They are nowadays the most important tool to detect Lefschetz properties in AG algebras when the Macaulay dual generator in known. We give also some applications in geometry and in algebra (see also [DGI, GZ]).

In [GZ] we introduced a family of Artinian Gorenstein algebras, whose combinatorial structure imposes that they are presented by quadrics. Some of these algebras have non-unimodal Hilbert vector. In particular we provide families of counter-examples to the conjecture that Artinian Gorenstein algebras presented by quadrics should satisfy the weak Lefschetz property.

In [DGI] we introduced higher order Jacobian ideals higher order polar maps, and higher order Milnor algebras associated to a reduced projective hypersurface. We use higher order Hessians and mixed order Hessians to study these higher order objects relating them to some standard graded Artinian Gorenstein algebras. We study the corresponding Hilbert functions and Lefschetz properties.

Higher Hessians

I recall that any standard graded Artinian Gorenstein $\mathbb{K}$-algebra of codimension n and socle degree d has a presentation

$$A = \mathbb{K}[X_1, \ldots, X_n]/Ann(f)$$

with $f \in R_d$ and $I_1 = 0$. Here we are considering $Q = \mathbb{K}[X_1, \ldots, X_n]$ as a ring of differential operators over the polynomial ring R.

Maeno and Watanabe introduced higher Hessians of order k (see [MW]). Let us recall the construction. Let $\mathcal{B} = \{\alpha_1, \dots, \alpha_s\}$ be a ordered basis of A_k.

Definition

The k-th Hessian matrix of f with respect to $\mathcal{B}$ is $\mathrm{Hess}_f^k = [\alpha_i\alpha_j(f)]$. The k-th Hessian is $\mathrm{hess}_f^k = \det \mathrm{Hess}_f^k$.

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Since the rank of the order k Hessian does not depend on the choice of ordered basis, we will omit the dependence on the basis. In the sequel we will see that for our purposes only the rank of the Hessian is needed.

Example

Let $f = x^5 + y^5 + z^5 \in \mathbb{C}[x, y, z]$ and let $A = \mathbb{C}[X, Y, Z]/\operatorname{Ann}_f$. Then $A_2 = < X^2, Y^2, Z^2 >$. Therefore:

$$\operatorname{Hess}_f^2 = 120 \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}.$$

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Theorem (Maeno-Watababe)

Let A be an AG algebra of socle degree $d \geq 2k$ and $L \in A_1$. Then the multiplication map

$$\bullet L^{d-2k} : A_k \to A_{d-k}$$

is an isomorphism if and only if $\mathrm{hess}_f^k(L^\perp) \neq 0$. *In particular, A has the SLP if and only if* $\mathrm{hess}_f^k \neq 0$ *for* $k = 1, \ldots, \lfloor \frac{d}{2} \rfloor$.

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Example (Ikeda's example simplified)

Let $f = xu^3v + yuv^3 \in \mathbb{K}[x, y, u, v]_5$ and let A be the associated Artinian Gorenstein algebra. The Hilbert function of A is

$$\mathrm{Hilb}(A) = (1, 4, 7, 7, 4, 1).$$

Since $X = \mathbb{P}^3$ is not a cone, then $\mathrm{hess}_f \neq 0$ by Gordan Noether theory. On the other side, $\mathrm{hess}_f^2 = 0$, therefore A fails WLP by Maeno-Watanabe Hessian criterion.

Corollary

Let $A = A_f$ be astandard graded Artinian Gorenstein $\mathbb{K}$-algebra of codimension n and socle degree d. Then:

1. *A has the SLP if and only if $\mathrm{hess}_f^k \neq 0$ for all $k \leq d/2$;*
2. *If $d \leq 4$, then A has the SLP if and only if $\mathrm{hess}_f \neq 0$;*
3. *If $d \leq 4$ and $n \leq 4$, then A has the SLP;*
4. *If $n \geq 5$ and $d \in \{3, 4\}$, then $\mathrm{hess}_f = 0$ if and only if A fails SLP.*

GNP polynomials

Let $N_i \in \mathbb{K}[x_1, \ldots, x_n]_k$ and let $M_i \in \mathbb{K}[u_1, \ldots, u_m]_e$ be monomials using all the variables.

Proposition (-)

Let $f = \sum_{i=1}^{s} N_i M_i$. *If* $s > \binom{m-1+k}{k}$, *then* $\text{hess}_f^k = 0$. *In particular* A_f *fails SLP in degree* k.

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Theorem (-)

Let $n, d \geq 3$ and let A be a standard graded Artinian Gorenstein $\mathbb{K}$-algebra. For each pair $(n, d) \notin \{(4,3),(4,4)\}$ and for every $k \leq d/2$ there exist standard graded Artinian Gorenstein algebras A of codimension $n \geq 4$ and socle degree d failing SLP in degree k.

Mixed Hessians

Now we want to introduce mixed Hessians. While the higher Hessians give a criterion to the failure of the SLP, the mixed Hessians give a criterion also for the WLP. Moreover, using mixed Hessians we will be able to express the multiplication map for a power of a linear form in an AG algebra instead of only capture its rank.

Let $k \leq l$ and suppose that $k + l \leq d$. Let let $B_k = \{\alpha_1, \ldots, \alpha_s\}$ and $B_l = \{\beta_1, \ldots, \beta_t\}$ be ordered $\mathbb{K}$-linear basis for A_k and A_l.

Definition

The mixed Hessian matrix of f of order (l, k) is

$$\mathrm{Hess}^{(l,k)}(f) := [\beta_i \alpha_j(f)].$$

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We recall again that the rank of the mixed Hessian matrix does not depend on the choice of ordered basis. For this reason we will omit the dependence on the basis.

For our purposes there will be useful to consider a slight modification in the mixed Hessian matrix, we will introduce the dual mixed Hessian matrix.

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Let $k \leq l$ and let $B_k = \{\alpha_1, \ldots, \alpha_s\}$ and $B_l = \{\beta_1, \ldots, \beta_t\}$ be ordered $\mathbb{K}$-linear basis for A_k and A_l. Let $B_l^* = \{\beta_1^*, \ldots, \beta_t^*\}$ be the dual basis with respect to the perfect pairing given by multiplication.

Definition

The dual mixed Hessian matrix of f of order (l^*, k) is

$$\text{Hess}^{(l^*,k)}(f) := [\beta_i^* \alpha_j(f)].$$

The following is a generalization of Maeno - Watanabe's result.

Theorem (-,Zappalà)

With the previous notation, let M be the matrix associated to the map $\bullet L^{l-k} : A_k \to A_l$ with respect to the bases B_k and B_l. Then

$$M = (l-k)!\,\mathrm{Hess}_f^{(l^*,k)}(L^\perp).$$

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Remark

Notice that since the rank does not depends on the choice of basis, we have:

$$\mathrm{rk}(\mathsf{Hess}_f^{(l^*,k)}) = \mathrm{rk}(\mathsf{Hess}_f^{(d-l,k)})$$

Corollary

Let $A = Q/\operatorname{Ann}_f$ *be a standard graded Artinian Gorenstein algebra. Then* A *has the WLP if and only if either*

1. $d = 2q + 1$ *and* $\operatorname{hess}_f^k \neq 0$ *or*
2. $d = 2q$ *and* $\operatorname{Hess}_f^{q-1,q}$ *does not have maximal rank.*

GNP polynomials (bis)

Proposition (-, Zappalà)

Let $f = \sum_{i=1}^{s} N_i M_i$. *If* $s > \binom{m-1+e-1}{e-1}$, *then* $\text{hess}_f^{(k,e-1)} = 0$. *In other words, the map* $\bullet L : A_k \to A_{k+1}$ *is not injective for any* $L \in A_1$.

Theorem (-, Zappalà)

For each pair $(n, d) \notin \{(4,3), (4,4), (5,4), (4,6)\}$ *with* $n \geq 4$ *and with* $d \geq 3$ *there exist* A *of codimension* n *and socle degree* d*, with a unimodal Hilbert vector failing WLP.*

AG algebras presented by quadrics

Migliori and Nagel conjectured that any Artinian Gorenstein algebras presented by quadrics should satisfy WLP (see [MN1, MN2]).

From the classification of cubics with vanishing Hessian for $N \leq 6$ there are no cubics forms f with vanishing Hessian such that $Ann(f)$ is generated by quadratic forms.

The first counterexample to NM conjecture is the following:

Example (-, Russo)

A tangent section of the secant variety of the Segre variety $\mathrm{Seg}(\mathbb{P}^2, \mathbb{P}^2) \subset \mathbb{P}^8$.

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A tangent section of the secant variety of the Segre variety $\mathrm{Seg}(\mathbb{P}^2, \mathbb{P}^2) \subset \mathbb{P}^8$.

Theorem (-, Zappalà)

For any $d \geq 5$ and $N >> 0$ there are Artinian Gorenstein algebras presented by quadrics whose Hilbert vector are totally not unimodal, that is

$$\dim A_1 > \dim A_2 > \ldots > \dim A_{\lfloor \frac{d}{2} \rfloor}.$$

Theorem (-,Zappalà)

There exists standard graded Artin Gorenstein algebras presented by quadrics of socle degree $d = 2k + 1 \geq 3$ *failing WLP for all* $\operatorname{codim} A \geq d + 5$.

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There exists a standard graded Artinian Gorenstein algebras presented by quadrics of socle degree $d = 2k \geq 4$ failing WLP for all $\operatorname{codim} A \geq d + 12$.

Higher order Jacobians, polar maps and Milnor algebras

We now construct higher order versions of these classical objects, explicit some relations among them and extend some classical results to this higher order context.

Higher order Jacobian

As usual, let $f \in R_d$ be a reduced polynomial and $A = Q/_f$ be the associated AG algebra. Let us consider the k-th order Jacobian ideal of $f \in R$ to be $J^k = J^k(f) = (Q_k * f) = (A_k * f)$, the ideal generated by the k-th order partial derivatives of f.

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There is a natural short exact sequence:

$$0 \to I_k \to Q_k \to J^K_{d-k} \to 0.$$

The map $\lambda : Q_k \to J^k_{d-k}$ is just $\lambda(\alpha) = \alpha * f$, given by the natural action of Q in R. Therefore, $\dim_{\mathbb{K}} J^k_{d-k} = \dim_{\mathbb{K}} A_k = h_k$.

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Higher order polar map

Definition

The k-th polar mapping (or k-th gradient mapping) of the hypersurface $X = V(f) \subset \mathbb{P}^n$ is the rational map $\Phi_X^k : \mathbb{P}^n \dashrightarrow \mathbb{P}^{\binom{n+k}{k}-1}$ given by the k-th partial derivatives of f. The k-th polar image of X is $\tilde{Z}_k = \overline{\Phi_X^k(\mathbb{P}^n)} \subseteq^{\binom{n+k}{k}-1}$, the closure of the image of the k-th polar map. Let $\{\alpha_1, \ldots, \alpha_{a_k}\}$ be a $\mathbb{K}$-linear basis of A_k, we define the relative k-th polar map of X to be the map $\varphi_X^k :^n \dashrightarrow^{a_k-1}$ given by the linear system J_{d-k}^k:

$$\varphi_X^k(p) = (\alpha_1(f)(p) : \ldots : \alpha_{a_k}(f)(p)).$$

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Theorem (Dimca, -, Ilardi)

With the above notations, we get:

$$\dim Z_k = \dim \tilde{Z}_k = \mathrm{rk}(\mathrm{Hess}_X^{(1,k)}) - 1.$$

In particular, the following conditions are equivalents:

(i) φ^k *is a degenerated map, that is,* $\dim Z_k < n$*;*

(ii) $\mathrm{rk}(\mathrm{Hess}_X^{(1,k)}) < n + 1$*;*

(iii) *The map* $\bullet L^{d-k-1} : A_1 \to A_{d-k}$ *has not maximal rank for any* $L \in A_1$*.*

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(iii) *The map* $\bullet L^{d-k-1} : A_1 \to A_{d-k}$ *has not maximal rank for any* $L \in A_1$*.*

Proposition (Dimca, -, Ilardi)

Let f be a homogeneous form of degree d and let $1 < k < d-1$. Consider the k-th polar map

$$\Phi_f^k : \mathbb{P}^N \dashrightarrow Z_k \subsetneq \mathbb{P}^{\binom{N+k}{k}-1}$$

given by the order k derivatives. If $\mathrm{hess}_f \neq 0$ then Φ_f^k is non degenerated, that is $\dim Z_k = N$.

Example

Let $X = V(f) \subset \mathbb{P}^3$ be the Ikeda surface, $f = xuv^3 + yu^3v + x^2y^3$. Let $A = Q/_f$, its Hilbert vector is $\mathrm{Hilb}(A) = (1, 4, 10, 10, 4, 1)$. Since $\mathrm{hess}_f \neq 0$, by GN Theorem, the second polar map:

$$\varphi_X^2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^9$$

is non degenerated.

k-smooth hypersurfaces

Let $X = V(f) \subset \mathbb{P}^n$ be a hypersurface. We say that X is k-smooth if every $p \in X$ has multiplicity $\geq k$.

Theorem (DGI)

Ik X is k-smooth, then φ_X^k is finite.

The higher order Milnor algebra

Let $A = Q/_f$ and J^k the k-th Jacobian. The k-th Milnor algebra associated to f is $M_f^k = R/J^k$.

Theorem (DGI)

The k-th Milnor algebra $M_f^k = R/J^k$ is Artinian if and only if X is k-smooth.

Higher order GN theory

1. Higher polar map/image.
2. Geometric description of non maximality of the rank of mixed Hessians.
3. SLP in codimension 3?
4. Exmples failing SLP/WLP for every codimension $n \geq 4$ and socle degree $d \geq 3$ with few exceptions.
5. Higher order GN Identity?
6. Classification in low codimension/degree?

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