On Hessians and the Lefschetz properties

Lecture 4: Hessians and the Jordan types for AG algebras

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Introduction

In this lecture we recall the concept of Jordan type for an Artinian algebra and use it to understand the level of failure of SLP and WLP for AG algebras. The main reference is [CG].

Jordan types

The Jordan type $\mathcal{J}_A$ of a standard graded Artinian Gorenstein $\mathbb{K}$-algebra

$A = \bigoplus_{i=0}^{d} A_i$ is the partition of $N = \dim_{\mathbb{K}} A$ given by the Jordan blocks of the multiplication map $\mu_l : A \to A$ for a generic linear form $l \in A_1$. Notice that this $\mathbb{K}$-linear map is nilpotent, so the Jordan blocks have only 0 in the diagonal. *A priori* the Jordan type depends on $l \in A_1$ but considering $\mathrm{char}(\mathbb{K}) = 0$ it is invariant in a open (Zariski) subset of A_1.

Let $A = \bigoplus_{i=0}^{d} A_i$ be a standard graded Artinian $\mathbb{K}$-algebra and let $M = \bigoplus_{j=0}^{m} M_j$ be a graded A-module. For $l \in A_1$ consider the map $\mu_l : M \to M$ given by $\mu_l(x) = lx$. Since $l^{d+1} = 0$ and $\mu_l^k = \mu_{l^k}$, we conclude that μ_l is a nilpotent $\mathbb{K}$-linear map whose eigenvalues are only 0. The Jordan decomposition of such a map is given by Jordan blocks with 0 in the diagonal, therefore it induces a partition of $\dim_{\mathbb{K}} M$ that we denote $\mathcal{J}_{M,l}$. Without loss of generality we consider the partition in a non decreasing order.

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Definition

Given a nilpotent linear homomorphism $\mu_l : M \to M$ there is a direct sum decomposition as $\mathbb{K}$-linear spaces $M = \bigoplus_{j=0}^{m} M_j$ into cyclic μ_l-invariant subspaces. We call a μ_l-invariant basis of these cyclic subspaces M_i, $< v_i, lv_i, \ldots, l^{k_i-1}v_i >$, a string of length $k_i = \dim M_i$. The partition $\mathcal{J}_{M,l}$ is given by the length of the strings $k_i = \dim_{\mathbb{K}} M_i$.

Partition

Definition

Given a partition $P = p_1 \oplus \ldots \oplus p_s$ of N with $p_1 \geq \ldots \geq p_s$, we denote $P^\vee$ the dual partition obtained from P exchange rows and columns in the Ferrer diagram (diagram of dots). If $P' = p'_1 \oplus \ldots \oplus p'_t$ is another partition of N with $p'_1 \geq \ldots \geq p'_t$, we say that P is less than or equal to P' ($P \preceq P'$) if either:

(i) $s < t$ or

(ii) $s = t$, $p_i = p'_i$ for $i = 1, \ldots, j-1$ and $p_j \leq p'_j$ for some $j \leq s = t$.

If the partition P have repeated terms, say $f_1, f_2, \ldots, f_r$ with multiplicity $e_1, e_2, \ldots, e_r$ respectively, in this case we write

$$P = f_1^{e_1} \oplus \ldots \oplus f_r^{e_r}.$$

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If $\mathbb{K}$ is a field of $\mathrm{char}(\mathbb{K}) = 0$, there is a Zariski open non empty subset of $\mathcal{U} \subset A_1$ where $\mathcal{J}_{M,l}$ is constant for $l \in \mathcal{U}$, we call it the Jordan type of M and we denote it $\mathcal{J}_M$. We are interested in the Jordan type of A as a module over itself, $\mathcal{J}_A$, in the case that A is a standard graded Artinian Gorenstein $\mathbb{K}$-algebra.

The following result is well known. It says that the WLP can be described by the number of blocks of a Jordan decomposition of μ_l.

Proposition

Suppose that A is a standard graded Artinian $\mathbb{K}$-algebra. Then A has the WLP if and only if the number of parts of $\mathcal{J}_A$ is equal to the maximum value of its Hilbert vector (called the Sperner number of A).

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The following Theorem shows that the Jordan type of A detects both WLP and SLP.

Theorem

Suppose that $A = \bigoplus_{i=0}^{d} A_i$ is a standard graded Artinian $\mathbb{K}$-algebra with $A_d \neq 0$ and let $l \in A_1$ for $k > 0$. Then:

1. *If $\mathrm{Hilb}(A)$ is unimodal, then $\mathcal{J}_{A,l} \preceq \mathrm{Hilb}(A)^\vee$;*
2. *The following conditions are equivalent:*
    1. *l is a Strong Lefschetz element;*
    2. *$\mathcal{J}_{A,l} = \mathrm{Hilb}(A)^\vee$.*

In particular, if A has the SLP, then $\mathcal{J}_A = \mathrm{Hilb}(A)^\vee$.

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For a given Artinian Gorenstein $\mathbb{K}$-algebra A and for any linear form $l \in A_1$, consider the short exact sequence:

$$0 \to lA_{i-1} \to A_i \to A_i/lA_{i-1} \to 0.$$

Since A_i is a finite dimensional $\mathbb{K}$-vector space, up to the choice of a basis completing a basis of lA_{i-1}, there is a linear subspace $\hat{A}_i \subset A_i$ such that $\hat{A}_i \simeq A_i/lA_{i-1}$.

Definition

Let us define

$$E_i^j = \{v \in \hat{A}_i | v l^j = 0 \text{ and } v l^{j-1} \neq 0\}, \ j = 1, \ldots, d - i.$$

denote $e_i^j = \dim E_i^j$, $e_j = \displaystyle\sum_{i=0}^{d} e_i^j$ for $j = 1, \ldots, d$ and $e_{d+1} = 1$.

Proposition

With the previous notation, the Jordan type of A is:

$$\mathcal{J}_A = \bigoplus_{j=1}^{d+1} j^{e_j}.$$

Lemma

Let r_i^j be teh rank of the generic multiplication map $\mu_l : A_i \to A_{i+j}$. Then

$$e_i^j = r_i^{j-1} - r_i^j - r_{i-1}^j + r_{i-1}^{j+1}.$$

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Theorem (Costa, -)

The Jordan type of a standard graded Artinian Gorenstein $\mathbb{K}$-algebra A depends only on the rank of the mixed Hessians. More precisely and explicitly,

$$\mathcal{J}_A = \bigoplus_{j=1}^{d+1} j^{e_j}$$

with $e_{d+1} = 1$ and for $j \leq d$, we have either:

(i) $e_j = 2\sum_{s=0}^{m} r_s^{j-1} - 4\sum_{s=0}^{m} r_s^{j} + 2\sum_{s=0}^{m-1} r_s^{j+1} + 2r_m^j$ *if* $d - j = 2m$ *or*

(ii) $e_j = 2\sum_{s=0}^{m} r_s^{j-1} - 4\sum_{s=0}^{m} r_s^{j} + 2\sum_{s=0}^{m} r_s^{j+1} + r_{m+1}^{j-1} - r_m^{j+1}$ *if* $d - j = 2m + 1.$

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String diagrams: The algorithm

Corollary

Let A be an standard graded Artinian Gorenstein $\mathbb{K}$-algebra with Hilbert vector $\mathrm{Hilb}(A)$*. The Jordan type of A can be obtained, algorithmically using Strings in a Ferrer diagram of* $\mathrm{Hilb}(A)$*.*

Jordan types in socle degree three

In this case $A = A_0 \oplus A_1 \oplus A_2 \oplus A_3$ and its Hilbert vector is $(1, n, n, 1)$. Let $r = rk\, \mathrm{Hess}_f$, then $\mathrm{d}r_1^1 = r \leq n$. By the Theorem 4 we get:

1. $e_1 = e_1^1 + e_2^1 = 2(n-r)$;
2. $e_2 = e_1^2 = r - 1$;
3. $e_3 = 0$;
4. $e_4 = 1$.

By definition $J_A = 4^1 \oplus 2^{r-1} \oplus 1^{2(n-r)}$.

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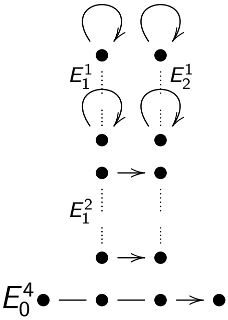

$$e_1^1 = e_2^1 = n - r$$

$$e_1^2 = r - 1$$

Figure: String diagram for algebras of socle degree three.

By Gordan-Noether Theorem, there is no hypersurfaces in $\mathbb{P}^{n-1}$ with $n \leq 4$ not a cone and having vanishing Hessian.
The classification of cubics with vanishing Hessian in low dimension is part of the classical work of Perazzo (see [Pe]) this classical work was revisited in [GRu].

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Perazzo's cubic

Example

The first example considered by Perazzo was $X = V(f) \subset \mathbb{P}^4$ with $f = xu^2 + yuv + zv^2$. Up to projective transformations it is the only cubic hypersurface with vanishing Hessian not a cone in $\mathbb{P}^4$ (see [GRu, Theorem 5.2]). The algebra $A = Q/\operatorname{Ann}_f$ has Hilbert vector $\operatorname{Hilb}(A) = (1, 5, 5, 1)$ and Jordan type $\mathcal{J}_A = 4^1 \oplus 2^3 \oplus 1^2 \prec 4^1 \oplus 2^5$ and $\Delta(A) = 1$.

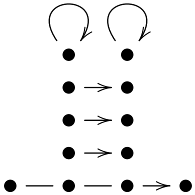

Figure: String diagram for $\mathcal{J}_A = 4^1 \oplus 2^3 \oplus 1^2$.

The cubic hypersurfaces $X = V(f) \subset \mathbb{P}^{n-1}$ with $n = 5, 6, 7$ not a cone and having vanishing Hessian were classified by Perazzo and the co-rank of Hess_f is 1 for all of them (see [Pe] or [GRu, Theorem 5.2, 5.3, 5.6, 5.7]).

We present now explicit examples of cubics with vanishing Hessian whose co-rank of Hess_f is grater than 1. A variation of such examples can be found in [GRu].

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Example

For $n = 8$, consider $f = xu^2 + yuv + zv^2$, f' a copy of f and $g = f \# f'$.

$$g = x_1 u^2 + x_2 uv + x_3 v^2 + x_4 vw + x_5 w^2$$

The co-rank of Hess_g is two. Putting $g_i = \frac{\partial g}{\partial x_i}$, the explicit relations among the derivatives are $g_1 g_3 = g_2^2$ and $g_3 g_5 = g_4^2$. It is easy to check that $V(g)$ is not a cone. Let $A = Q/\,\mathrm{Ann})g$, then:

$$\mathrm{Hilb}(A) = (1, 8, 8, 1)$$

$$\mathcal{J}_A = 4^1 \oplus 2^5 \oplus 1^4 \prec 4^1 \oplus 2^6 \oplus 1^2 \prec \mathrm{Hilb}(A)^\vee.$$

Therefore, $\Delta(A) = 2$. Its String diagram is:

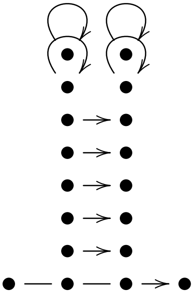

Figure: String diagram for $\mathcal{J}_A = 4^1 \oplus 2^5 \oplus 1^4$.

Example

For $n = 9$, consider

$$f = x_1 u_1^2 + x_2 u_1 u_2 + x_3 u_2^2 + x_4 u_2 u_3 + x_5 u_3^2 + x_6 u_3 u_1.$$

The co-rank of Hess_f is 3. Putting $f_i = \frac{\partial f}{\partial x_i}$, the explicit relations among the derivatives are $f_1 f_3 = f_2^2$, $f_3 f_5 = f_4^2$ and $f_5 f_1 = f_6^2$ and they are algebraically independent. It is easy to check that $V(f)$ is not a cone. Let $A = Q/\mathrm{Ann}_f$, then:

$$\mathrm{Hilb}(A) = (1, 9, 9, 1)$$

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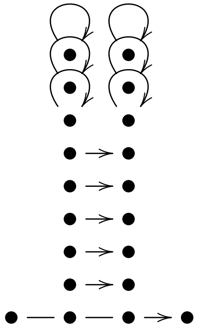

Figure: String diagram for $\mathcal{J}_A = 4^1 \oplus 2^5 \oplus 1^6$.

Theorem

Let A be a standard graded Artinian Gorenstein $\mathbb{K}$-algebra of Hilbert vector $\mathrm{Hilb}(A) = (1, n, n, 1)$. *Then the possibles Jordan types of A are:*

(i) *For $n \leq 4$, A has the SLP, therefore $\mathcal{J}_A = 4^1 \oplus 2^{n-1}$, i.e. $\Delta(A) = 0$;*

(ii) *For $n \in \{5, 6, 7\}$, either A has the SLP and $\mathcal{J}_A = 4^1 \oplus 2^{n-1}$ or A fails the SLP and $\mathcal{J}_A = 4^1 \oplus 2^{n-2} \oplus 1^2$, i.e. $\Delta(A) \leq 1$;*

(iii) *For $n = 8$, either A has the SLP and $\mathcal{J}_A = 4^1 \oplus 2^7$ or A fails the SLP and $\mathcal{J}_A = 4^1 \oplus 2^6 \oplus 1^2$ or $\mathcal{J}_A = 4^1 \oplus 2^5 \oplus 1^4$, i.e. $\Delta(A) \leq 2$;*

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Let A be a standard graded Artinian Gorenstein $\mathbb{K}$-algebra of Hilbert vector $\mathrm{Hilb}(A) = (1, n, n, 1)$. *Then the possibles Jordan types of A are:*

(i) *For $n \leq 4$, A has the SLP, therefore $\mathfrak{J}_A = 4^1 \oplus 2^{n-1}$, i.e. $\Delta(A) = 0$;*

(ii) *For $n \in \{5, 6, 7\}$, either A has the SLP and $\mathfrak{J}_A = 4^1 \oplus 2^{n-1}$ or A fails the SLP and $\mathfrak{J}_A = 4^1 \oplus 2^{n-2} \oplus 1^2$, i.e. $\Delta(A) \leq 1$;*

(iii) *For $n = 8$, either A has the SLP and $\mathfrak{J}_A = 4^1 \oplus 2^7$ or A fails the SLP and $\mathfrak{J}_A = 4^1 \oplus 2^6 \oplus 1^2$ or $\mathfrak{J}_A = 4^1 \oplus 2^5 \oplus 1^4$, i.e. $\Delta(A) \leq 2$;*

(iv) *For $n = 9$, either A has the SLP and $\mathfrak{J}_A = 4^1 \oplus 2^8$ or A fails the SLP and $\mathfrak{J}_A = 4^1 \oplus 2^7 \oplus 1^2$ or $\mathfrak{J}_A = 4^1 \oplus 2^6 \oplus 1^4$ or $\mathfrak{J}_A = 4^1 \oplus 2^5 \oplus 1^6$, i.e. $\Delta(A) \leq 3$;*

(v) *For any positive integer δ, there are n and A such that*

Jordan type in socle degree four

Proposition

The Jordan type of an algebra $A = Q/\operatorname{Ann}_f$ *with Hilbert vector* $(1, n, a, n, 1)$ *such that* $\operatorname{rk} Hess_f = r \leq n$ *and*

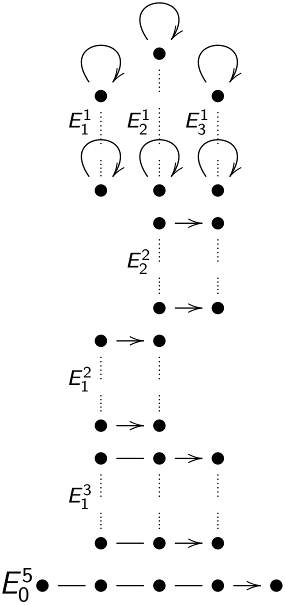
$$\operatorname{rk} Hess_f^{(1,2)} = s \leq n$$

is

$$\mathcal{J}_A = 5^1 \oplus 3^{r-1} \oplus 2^{2(n-r)} \oplus 1^{a-2n+r}.$$

By the Theorem 4 we get:

1. $e_1 = e_1^1 + e_2^1 + e_3^1 = 2n + a - 4s + r;$
2. $e_2 = e_1^2 + e_2^2 = 2(s - r);$
3. $e_3 = e_1^3 = r - 1;$
4. $e_4 = 0;$
5. $e_5 = 1.$



$$e_1^1 = e_3^1 = n - s$$
$$e_2^1 = a - 2s + r$$
$$e_2^2 = s - r$$
$$e_1^2 = s - r$$
$$e_1^3 = r - 1$$

Figure: String diagram for socle degree four

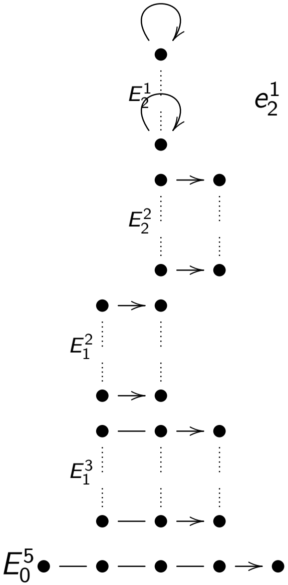
Corollary

The Jordan type of an algebra $A = Q/\operatorname{Ann}_f$ having the WLP with Hilbert vector $(1, n, a, n, 1)$ such that $\operatorname{rk} Hess_f = r \leq n$ is

$$\mathcal{J}_A = 5^1 \oplus 3^{r-1} \oplus 2^{2(n-r)} \oplus 1^{a-2n+r}.$$

By Theorem 4 we get:

1. $e_1 = e_1^1 + e_2^1 + e_3^1 = 0 + a - 2n + r$;
2. $e_2 = e_1^2 + e_2^2 = (n - r) + (n - r) = 2(n - r)$;
3. $e_3 = e_1^3 = r - 1$;
4. $e_4 = 0$;
5. $e_5 = e_0^5 = 1$.



$$e_2^1 = a - 2n + r$$

$$e_2^2 = n - r$$

$$e_1^2 = n - r$$

$$e_1^3 = r - 1$$

Figure: String diagram for socle degree four with WLP.

Example

For $n = 8$, consider $A = Q/\operatorname{Ann}_f$ with

$$f = x_1 u^2 v + x_2 u v^2 + x_3 u^3 + x_4 u w^2 + x_5 u^2 w.$$

It is easy to see that $\operatorname{Hilb}(A) = (1, 8, 10, 8, 1)$ and that A has the WLP. On the other hand $r = \operatorname{rk} \operatorname{Hess}_f = 6$. In fact putting $f_i = \frac{\partial f}{\partial x_i}$, we get the following explicit relations among the derivatives $f_2 f_3 = f_1^2$ and $f_3 f_4 = f_5^2$. Therefore:

$$\mathcal{J}_A = 5^1 \oplus 3^5 \oplus 2^4 \prec 5^1 \oplus 3^6 \oplus 2^2 \oplus 1^1 \prec \operatorname{Hilb}(A)^\vee, \ \Delta(A) = 2.$$

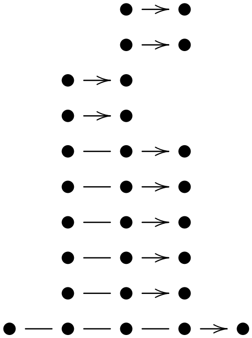


Figure: String diagram for $\mathcal{J}_A = 5^1 \oplus 3^5 \oplus 2^4 \prec 5^1 \oplus 3^6 \oplus 2^2 \oplus 1^1$.

Theorem

Let A be a standard graded Artinian Gorenstein $\mathbb{K}$-algebra of Hilbert vector $\mathrm{Hilb}(A) = (1, n, a, n, 1)$. *Then the possibles Jordan types of A are:*

(i) *If $n \leq 4$, then A has the SLP, therefore* $\mathcal{J}_A = 5^1 \oplus 3^{n-1} \oplus 1^{a-n}\,\mathrm{Hilb}(A)^\vee$, *i.e.,* $\Delta(A) = 0$;

(ii) *If $n = 5$, then either A has the SLP and* $\mathcal{J}_A = 5^1 \oplus 3^{n-1} \oplus 1^{a-n} = \mathrm{Hilb}(A)^\vee$ *or A fails SLP but has the WLP and* $\mathcal{J}_A = 5^1 \oplus 3^{n-2} \oplus 2^2 \oplus 1^{a-n-1}$, *i.e.,* $\Delta(A) \leq 1$;

(iii) *If $n \geq 6$, then there are algebras failing WLP.*

(iv) *For any positive integer δ, there are n and A such that* $\Delta(A) = \delta$.

Jordan types in socle degree five

Proposition

The Jordan type of an algebra $A = Q/\operatorname{Ann}_f$ *with unimodal Hilbert vector* $(1, n, a, a, n, 1)$ *such that* $\operatorname{rk} Hess_f = r_1^3 = r \leq n$, $\operatorname{rk} Hess_f^{(1,2)} = r_1^2 = p \leq n$, $\operatorname{rk} Hess_f^2 = r_2^1 = q$ *and* $\operatorname{rk} Hess_f^{(1,3)} = r_1^1 = s$ *is*

$$\mathcal{J}_A = 6^1 \oplus 4^{r-1} \oplus 3^{2(p-r)} \oplus 2^{2s-4p+q+r} \oplus 1^{2n+2a-4s-2q+2p}.$$

By Theorem 4 we get:

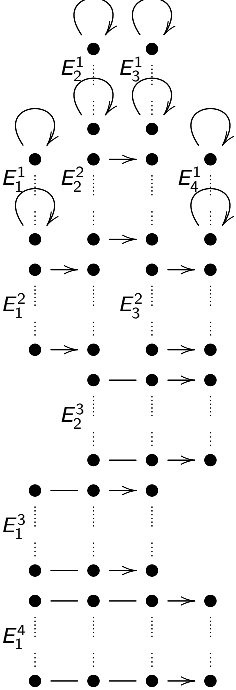
1. $e_1 = e_1^1 + e_2^1 + e_3^1 + e_4^1 = 2n + 2a - 4s - 2q + 2p$;
2. $e_2 = e_1^2 + e_2^2 + e_3^2 = 2s - 4p + q + r$;
3. $e_3 = e_1^3 + e_2^3 = 2(p - r)$;
4. $e_4 = e_1^4 = r - 1$;
5. $e_5 = 0$;
6. $e_6 = e_0^6 = 1$.

We have
$\mathcal{J}_A = 6^1 \oplus 4^{r-1} \oplus 3^{2(p-r)} \oplus 2^{2s-4p+q+r} \oplus 1^{2n+2a-4s-2q+2p}$.

By Theorem 4 we get:

1. $e_1 = e_1^1 + e_2^1 + e_3^1 + e_4^1 = 2n + 2a - 4s - 2q + 2p$;
2. $e_2 = e_1^2 + e_2^2 + e_3^2 = 2s - 4p + q + r$;
3. $e_3 = e_1^3 + e_2^3 = 2(p - r)$;
4. $e_4 = e_1^4 = r - 1$;
5. $e_5 = 0$;
6. $e_6 = e_0^6 = 1$.

We have
$\mathcal{J}_A = 6^1 \oplus 4^{r-1} \oplus 3^{2(p-r)} \oplus 2^{2s-4p+q+r} \oplus 1^{2n+2a-4s-2q+2p}$.



$$e_2^1 = e_3^1 = a - s - q + p$$

$$e_1^1 = e_2^2 = e_4^1 = n - s$$

$$e_1^2 = e_3^2 = s - p$$

$$e_2^3 = p - r$$

$$e_1^3 = p - r$$

$$e_1^4 = r - 1$$

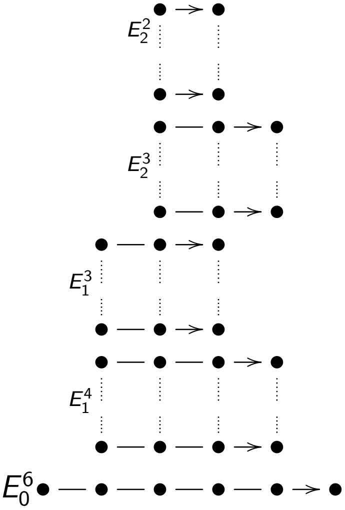
Corolário

The Jordan type of an algebra $A = Q/\operatorname{Ann}_f$ *having the WLP with Hilbert vector* $(1, n, a, a, n, 1)$ *such that* $\operatorname{rk} Hess_f = r \leq n$ *is*

$$\mathcal{J}_A = 6^1 \oplus 4^{r-1} \oplus 3^{2(n-r)} \oplus 2^{a-2n+r}.$$

Let $A = Q/\operatorname{Ann}_f$ with Hilbert vector $H_A = (1, n, a, a, n, 1)$. The WLP condition implies $a \geq n$, see Remark **??** and $r_1^1 = r_3^1 = r_1^2 = r_2^2 = n$, $r_2^1 = a$. The result follows from Proposition 10.

The result also can be obtained from Proposition 1 in the same way we did in Corollary 7.



$$e_2^2 = a + r - 2n$$

$$e_2^3 = n - r$$

$$e_1^3 = n - r$$

$$e_1^4 = r - 1$$

Figure: String diagram for socle degree five with WLP.

Example

The next example is a simplification of an example given by H. Ikeda (see [?, ?]). Both examples have the same type pattern of vanishing of Hessians, but this one has a simpler Hilbert vector.

Consider the algebra $A = Q/\operatorname{Ann}_f$ with

$$f = xu^3v + yuv^3$$

$$\operatorname{Hilb}(A) = (1, 4, 7, 7, 4, 1)$$

It is easy to see that $\operatorname{hess}_f \neq 0$ but $\operatorname{hess}_f^2 = 0$. Hence

$$\mathcal{J}_A = 6^1 \oplus 3^4 \oplus 2^2 \oplus 1^2 \prec \operatorname{Hilb}(A)^\vee, \ \Delta(A) = 1.$$

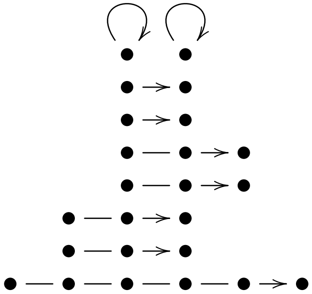


Figure: String diagram for $\mathcal{J}_A = 6^1 \oplus 3^4 \oplus 2^2 \oplus 1^2$.

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