FREE BOUNDARY CONSTANT MEAN CURVATURE HYPERSURFACES

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Lecture 1

CIMPA/ICTP RESEARCH IN PAIRS

Supported also by CNPq Grant 403349/2021-4

June 12, 2023

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4. Lecture 4
   - Some characterization of the critical catenoid. Index.

Lecture 1

1. Introduction/Motivation
2. Minimal and CMC surfaces
3. Stability result

Figure: Carl Friedrich Gauss (1777-1855)

Figure: Bernhard Riemann (1826 - 1866)

Theorem (Isoperimetric problem)

Let C be a simple and closed curve. Denote the area enclosed by A and L the length of C. Then

$$L^2 - 4\pi A \geq 0,$$

with equality if, and only if, C is a circle.

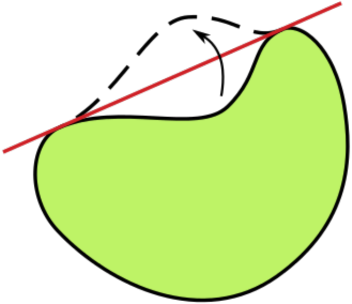
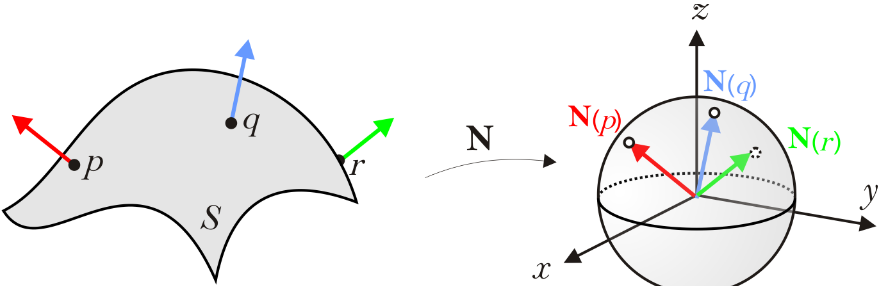


Figure: Image by Wikipedia



p
q
r
S
$\mathbf{N}$
z
$\mathbf{N}(q)$
$\mathbf{N}(p)$
$\mathbf{N}(r)$
y
x

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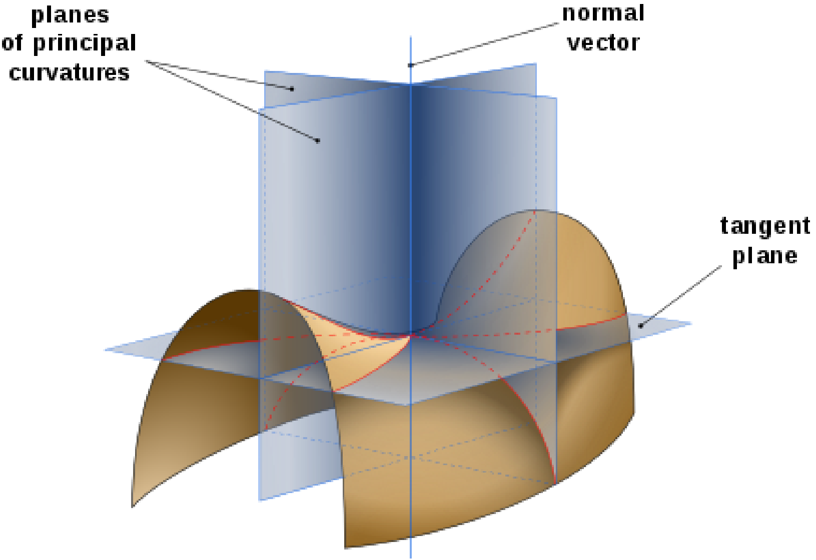
We can define the shape operator: $dN_p : T_pS \to T_pS$, which is self-adjoint operator. Consider an orthonormal basis $\{e_1, e_2\}$ on T_pS such that

$$\begin{aligned} dN_p e_1 &= -k_1 e_1 \\ dN_p e_2 &= -k_2 e_2 \end{aligned}$$

- We call k_1 and k_2 as the principal curvature. We define:
- Gaussian curvature $K = det(-dN) = k_1 k_2$.
- Mean curvature $H = \frac{1}{2}(k_1 + k_2)$.



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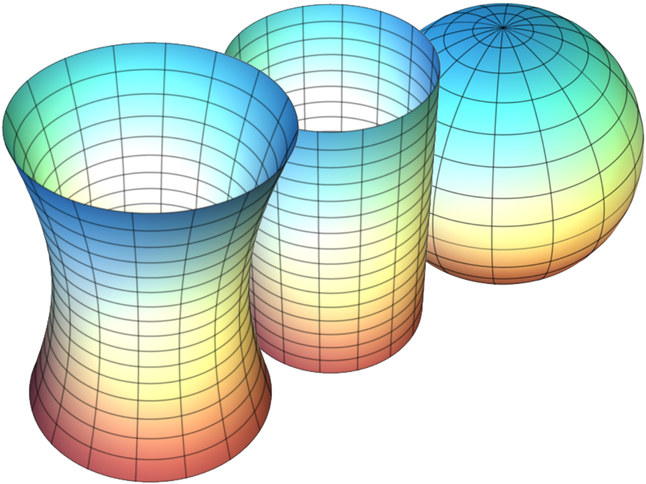

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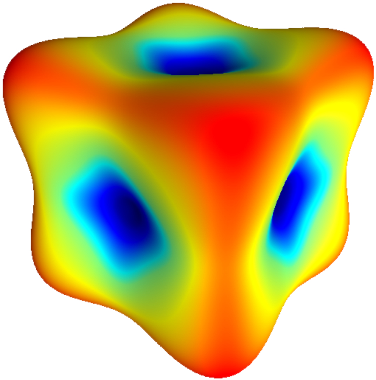


Figure: Mean Curvature.

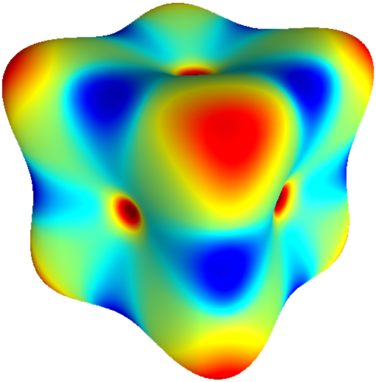


Figure: Gaussian Curvature.

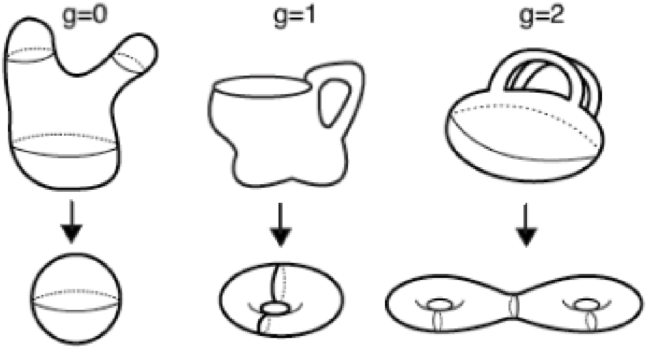


Figure: Image by Wikipedia.

[1] Some slides are courtesy by Professor Marcos Petrúcio Cavalcante.

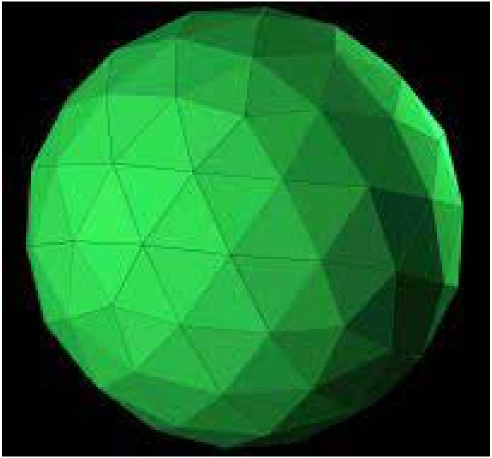


Figure: @http://www.salsburg.com/geod/geodesicmath.pdf

Consider a triangulation of S. We remember that the Euler characteristic of S is given by

$$\chi(S) = V - E + F,$$

where V, E, and F are respectively the numbers of vertices (corners), edges and faces in the given triangulation.

Moreover,

$$\chi(S) = 2 - 2g(S).$$

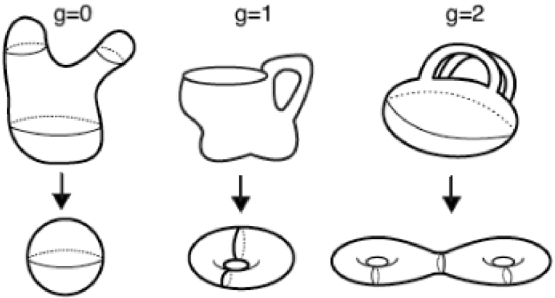


Figure: @Annenberg Learner

$$\chi(S) = 2 - 2g(S).$$

- $\chi(sphere) = 2,$
- $\chi(torus) = 0,$
- $\chi(bi - torus) = -2...$

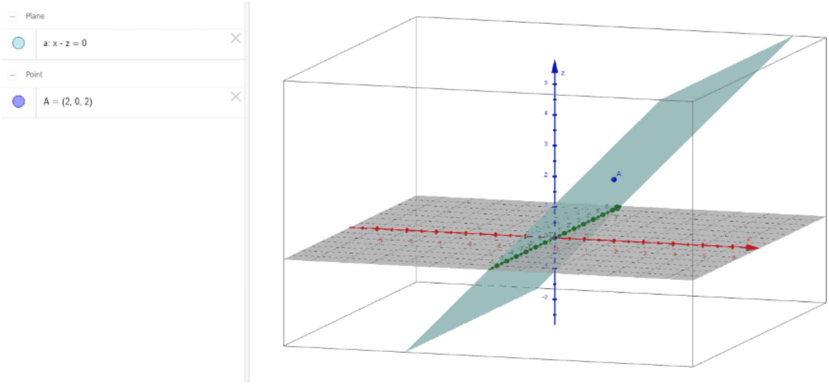
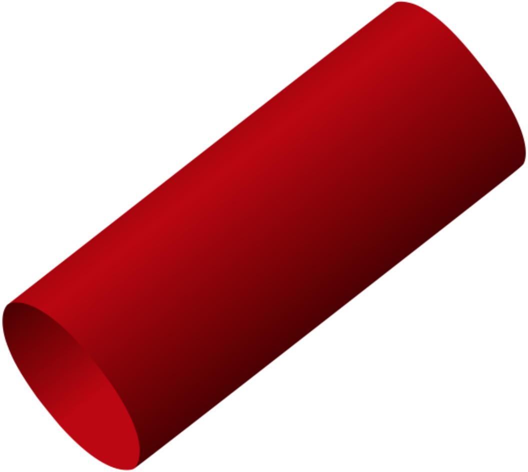


Figure: geogebra.org



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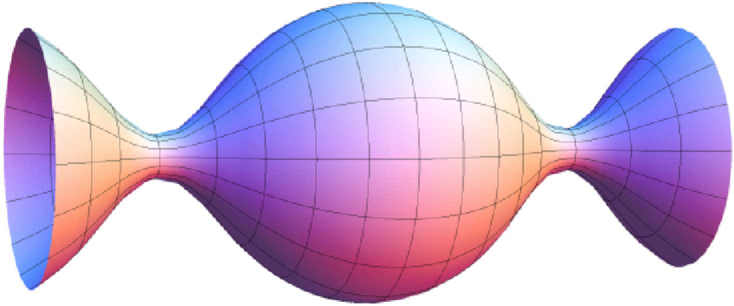

Figure: Image by Wikimedia

Theorem (Gauss-Bonnet Theorem)

For a closed surface S:

$$\int_S K dA = 2\pi\chi(S) = 2\pi(2-2g).$$

Figure: M. Spivak. A Comprehensive Introduction to Differential Geometry, Vol. 5, 3rd Edition.

In 1965, Willmore began the study about the functional defined on compact surfaces $M \subset \mathbb{R}^3$.

$$\mathcal{W}(M) = \int_M H^2 dM.$$

- It is invariant under conformal transformations of $\mathbb{R}^3$.

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Theorem (Willmore)

The Willmore energy satisfies $\mathcal{W}(M)$

$$\mathcal{W}(M) \geq 4\pi,$$

holds the equality if and only if M is an embedded round sphere.

Proof.

Use the blackboard.

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Willmore showed that round spheres have the least possible Willmore energy among all compact surfaces in three-space.[2]

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Willmore checked that $\mathcal{W}(M) \geq 2\pi^2$ in certain class of torus and the equality is achieved by the torus of revolution whose generating circle has radius 1 and center at distance from the axis of revolution:

$$(u, v) \mapsto (\sqrt{2} + cos(u))cos(v), (\sqrt{2} + cos(u)) \sin(v), \sin(u)) \in \mathbb{R}^3$$

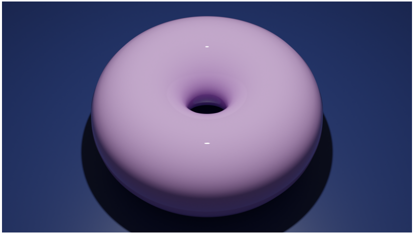


Figure: The torus with minimal Willmore energy, with major radius $\sqrt{2}$ and minor radius 1. Image by Wikipedia.

Conjecture (Willmore Conjecture - 1965)

[3] *Every compact surface M of genus one in $\mathbb{R}^3$ must satisfy*

$$\mathcal{W}(M) = \int_M H^2 dM \geq 2\pi^2$$

[3] It is proved by F. C. Marques and A. Neves in 2013 - Min-max theory and the Willmore conjecture". Annals of Mathematics. 179 (2014), no. 2, 683 - 782

Minimal and Constant Mean Curvature Surfaces

For each f with compact support in $M \subset \mathbb{R}^3$, we can define the normal variation families

$$M(t) = \{p + tf(p)N(p); p \in M\}, \ t \in (-\epsilon, \epsilon).$$

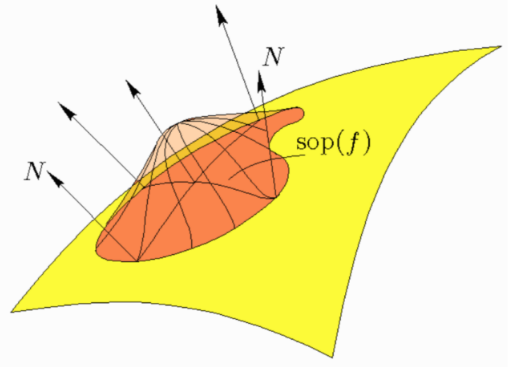


Figure: Image by www.ugr.es/ jperez

We define

$$A(t) = \mathsf{area}(M(t)),$$

we can prove that

$$A'(0) = -2\int_M f H dM.$$

Thus,

$$A'(0) = 0 \ \forall f \in C_0^\infty(M) \text{ if and only if, } H = 0.$$

We call that M is a minimal surface if H=0.

Remember the famous Plateau problem, which consist of finding a surface of least area bounding any given Jordan curves (was solved for Euclidean spaces in the 1930s independently by Douglas and Rado).

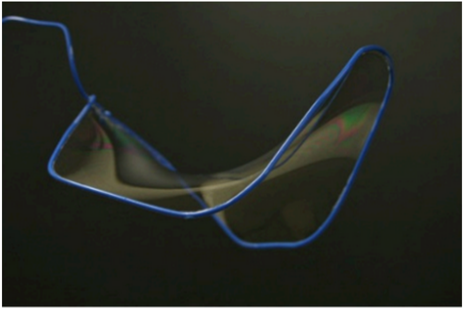
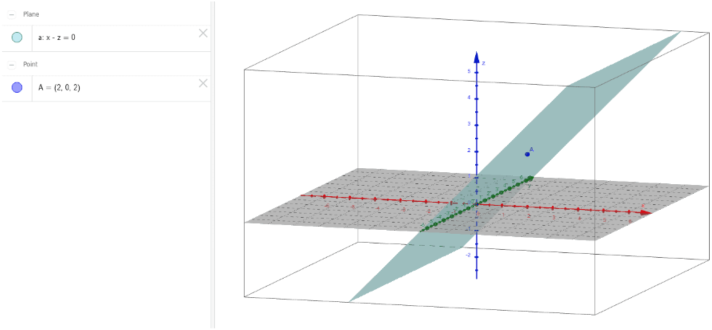


Figure: Plateau who experimented with soap films. www.math.hmc.edu/ jacobsen

Examples of minimal surfaces



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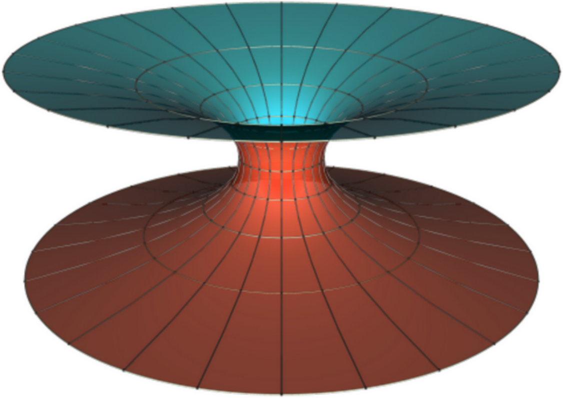


Figure: Image by indiana.edu/ minimal

Theorem

If M is a complete minimal surface of revolution, then M is a plane or a catenoid.

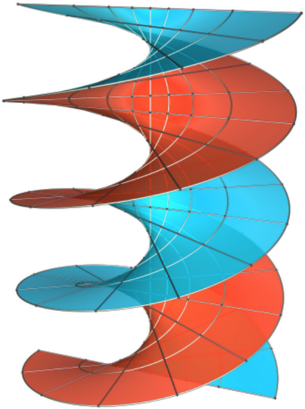


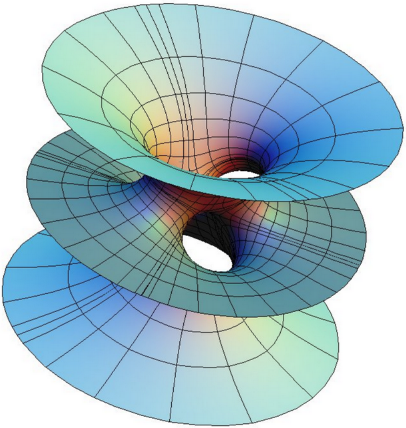
Figure: indiana.edu/minimal

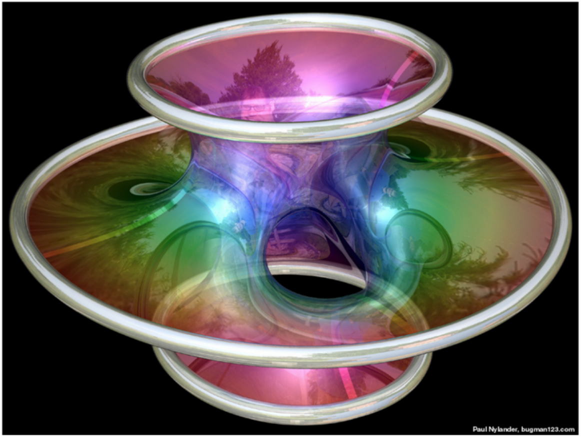
Theorem (Meeks-Rosenberg - 2005)

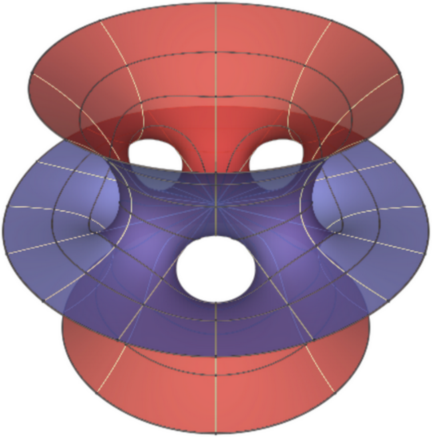
If M *is a embedded, simply connected*[4] *minimal surface, then* M *is a plane or a helicoid.*

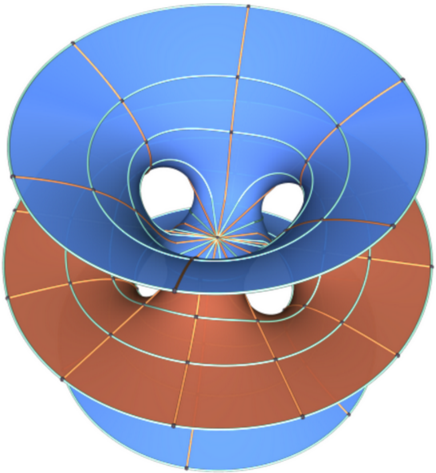
[4] If any simple closed curve can be shrunk to a point continuously in M.

Until its discovery, the plane, helicoid and the catenoid were believed to be the only embedded minimal surfaces that could be formed by puncturing a compact surface.




Paul Nylander, bugman123.com





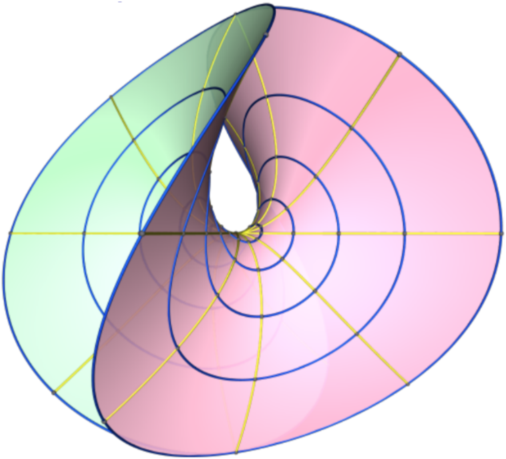
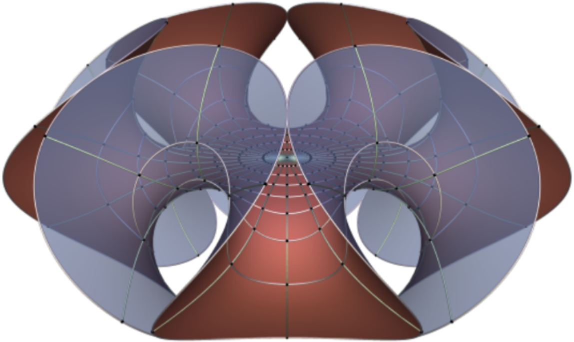


Figure: indiana.edu/minimal



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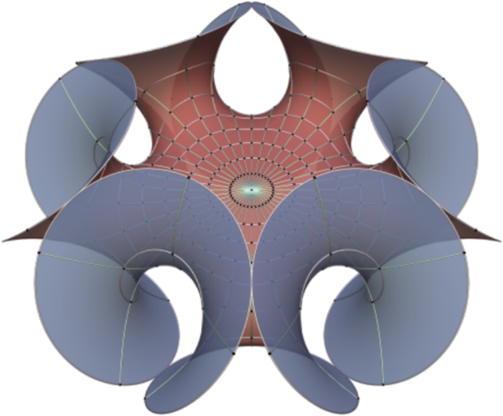

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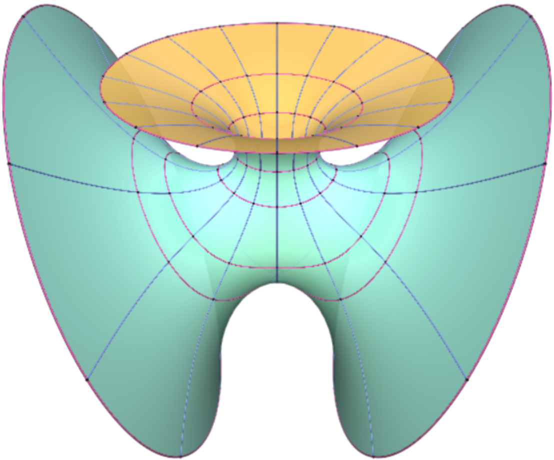
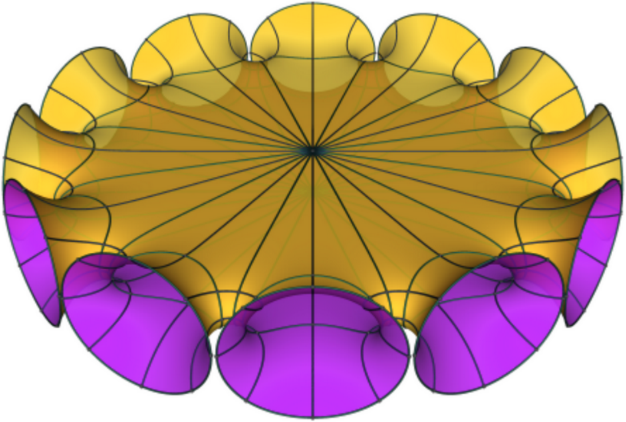




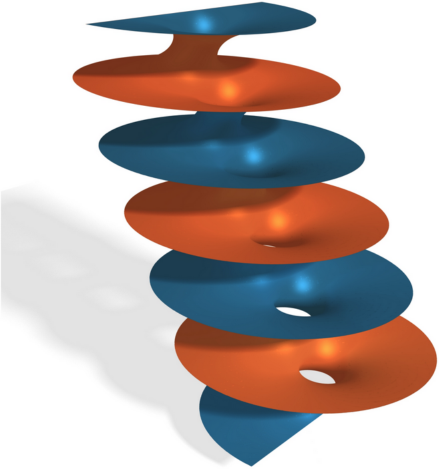
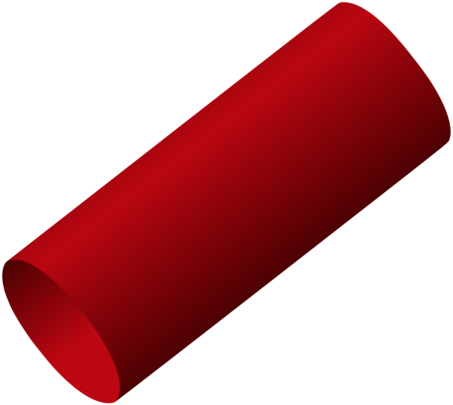


Figure: indiana.edu/minimal

Examples of CMC surfaces

Figure: www.maxpixel.net



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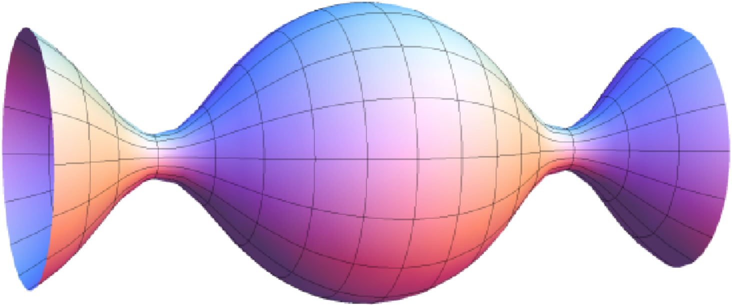

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- Hopf conjetured that the condition of topological 2-sphere can be removed.

Theorem (Alexandrov, 1956)

If M *is an embedded and closed CMC in* $\mathbb{R}^3$*, then* M *is a round sphere.*

- In 1982, Hsiang et al. constructed a CMC topological 3-sphere in $\mathbb{R}^4$, which is not round sphere.

Some results about stability of CMC or minimal surfaces



Classic isoperimetric problem: Among all the surfaces in $\mathbb{R}^3$ that bound a given volume, which is the surface with the smallest area?

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Until then, the only known example was the sphere.

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- In 1951, Hopf proved that the sphere is the only compact CMC surface of genus 0.
- Between 1956 and 1962, Alexandrov proved that the sphere is the only CMC, compact, embedded (does not have self-intersections) surface in $\mathbb{R}^3$.

In 1984, L. Barbosa and M. do Carmo proved the following theorem:

Theorem (Barbosa, Do Carmo, 1984)

Let M^n be a compact oriented n-manifold and let $x : M \to \mathbb{R}^{n+1}$ be an immersion with non-zero constant mean curvature. Then, x is stable if and only if $x(M^n) \subset \mathbb{R}^{n+1}$ is a round sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$.

- In 1986, Wente showed the existence of a compact surface immersed in $\mathbb{R}^3$ CMC that has the topology of a torus, thus providing a counterexample to Hopf's conjecture.

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- In 1983, for $n > 2$ Hsiang, Teng, Yu gave examples of compact non-spherical CMC hypersurfaces embedded in $\mathbb{R}^{n+1}$. These hypersurfaces and Wente's example are not stable by the theorem of Barbosa and Do Carmo.

If M is a minimal or CMC surface, then M is a critical point of the area function:

$$A'(0) = 0.$$

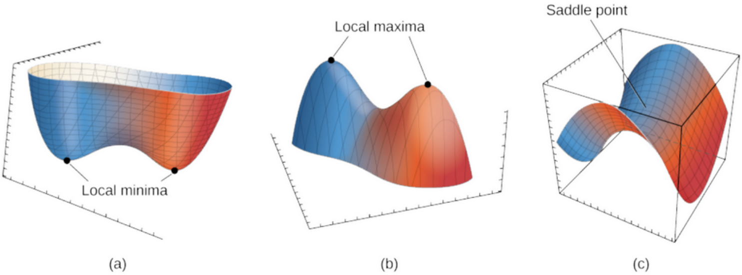

Figure: http://math.libretexts.org

Theorem (Second variation of area)

$$A''(0) = -\int_M f(\Delta f + |B|^2 f)dv =: Q(f, f).$$

Where,

$$|B|^2 = k_1^2 + k_2^2.$$

A surface $S \subset \mathbb{R}^3$ is called stable if

$$\frac{d^2}{dt^2} A(S_t)|_{t=0} \geq 0,$$

for all normal variation with compact support ($\int_S f = 0$).

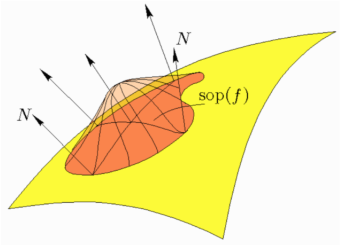

Figure: Image by www.ugr.es/ jperez

Barbosa - Do Carmo Theorem

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Lemma

1. *Let $x : S \to \mathbb{R}^3$ be a CMC immersion, then*

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$$\Delta|x|^2 = 4 + 4H\langle N, x\rangle.$$

3. *If* $x : S \to \mathbb{R}^3$ *is a compact umbilical immersion, then* S *is the sphere.*

Proof.

Let $f = 1 + H\langle N, x\rangle$, we have

$$\int_S f = 0,$$

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$$\begin{aligned}
\frac{d^2}{dt^2}A(S_t)|_{t=0} &= -\int_S f(\Delta_S f + |B|^2 f)dS \\
&= -\int_S (1 + Hh)(-2H^2 - |B|^2 Hh + |B|^2(1 + Hh))dS
\end{aligned}$$

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Thus,

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By hypothesys S is stable, then $\frac{d^2}{dt^2}A(S_t)|_{t=0} \geq 0.$

This implies that, $\frac{d^2}{dt^2}A(S_t)|_{t=0} = 0$. Thus, S is umbilical. By Lemma item 3, we conclude that S is a sphere. $\square$

Free boundary CMC surface in the ball

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Let $x : \Sigma^2 \to \mathbb{B}^3$ be an isometric immersion, where Σ is a smooth compact surface with $\Sigma \cap \partial\mathbb{B}^3 = \partial\Sigma$.

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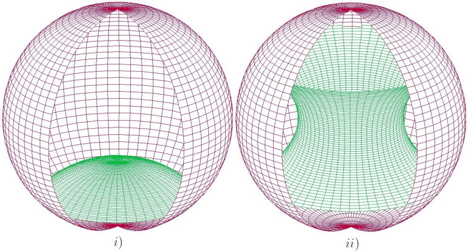

Figure: Image by Barbosa, Cavalcante and Pereira.

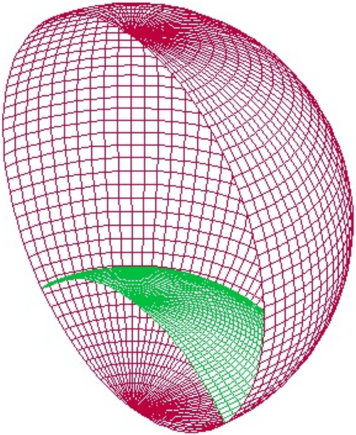

Figure: Spherical cap. Image by Barbosa, Cavalcante and Pereira.

Trivial: equatorial disc

$\mathbb{D}^2 = \{(x_1, x_2, 0) \in \mathbb{B}^3_1;\ x_1^2 + x_2^2 \leq 1\}$

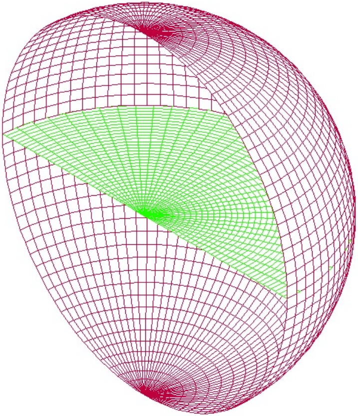

Figure: Image by Barbosa, Cavalcante and Pereira.

Critical catenoid

$\Sigma^2 = \{\big(a\cosh(\frac{t}{a})\cos(\theta), a\cosh(\frac{t}{a})\sin(\theta), t\big) \in \mathbb{B}^3_1\}$ where
$\theta \in [0, 2\pi], -ak_0 \leq t \leq ak_0$ and $k_0 > 0$ is the solution of $\cosh(k_0) = \frac{1}{k_0}$ e
$a = \frac{1}{\sqrt{\cosh^2(k_0) + k_0^2}}$

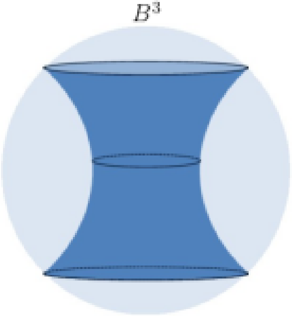

Figure: Image by Tayanara Santos.

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$$\frac{d}{dt}|_{t=0}|\Phi_t(M)| = -\int_\Sigma \langle X, H\rangle d\mu_\Sigma + \int_{\partial\Sigma} \langle X, \eta\rangle d\mu_{\partial\Sigma},$$

where

- η is the outward with unit conormal vector of $\partial\Sigma$.
- H is the mean curvature vector of Σ in M.
- $X = \dfrac{d\Phi}{dt}|_{t=0}$ is the variation field.
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- $\dfrac{d}{dt}|_{t=0}|\Phi_t(M)| = 0 \Longleftrightarrow H = 0$ and $\Sigma \perp \partial M$ along $\partial\Sigma$.
- The first variation formula shows that free boundary CMC surfaces are critical points of the area functional for volume preserving variations of Σ, whose $\partial\Sigma$ is free to move in ∂M.