FREE BOUNDARY CONSTANT MEAN CURVATURE HYPERSURFACES

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Lecture 2

CIMPA/ICTP RESEARCH IN PAIRS

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- This is a mini-course at master/beginning of PhD level.
  1. Some classic results.
  2. Free boundary CMC (minimal) (hyper)surfaces in the ball.
  3. Gaps results.
  4. Stability.
  5. Index.
  6. The Steklov Eigenvalue Problem.
  7. Some characterization of the critical catenoid.
  8. Some open problems.

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   - (Brief) Motivation to study Differential Geometry. (done)

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   - Free boundary CMC (hyper)surface in the ball. Stability.
4. Lecture 4
   - Some characterization of the critical catenoid. Index.

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- In 1986, Wente constructed a CMC immersion of 2-torus in $\mathbb{R}^3$, which disproves Hopfs conjecture. [1]

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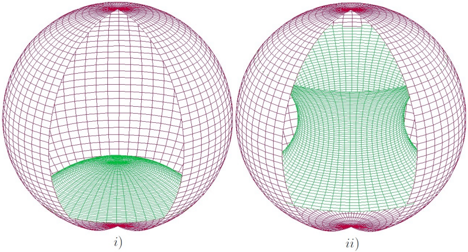


Figure: Image by Barbosa, Cavalcante and Pereira.

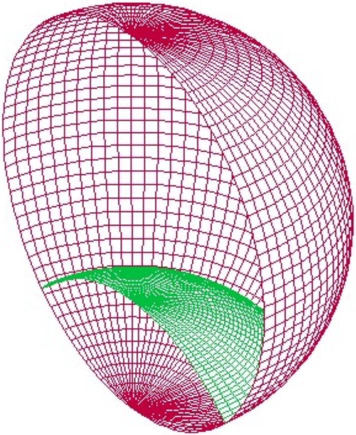
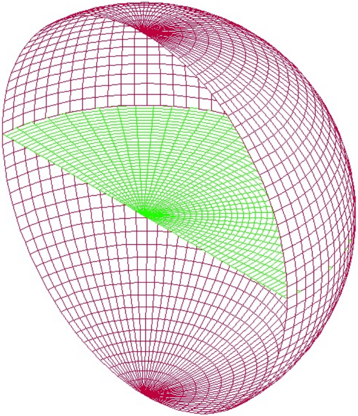


Figure: Spherical cap. Image by Barbosa, Cavalcante and Pereira.

Trivial: equatorial disc

$\mathbb{D}^2 = \{(x_1, x_2, 0) \in \mathbb{B}^3_1;\ x_1^2 + x_2^2 \leq 1\}$



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Critical catenoid

$\Sigma^2 = \{\big(a\cosh(\frac{t}{a})\cos(\theta), a\cosh(\frac{t}{a})\sin(\theta), t\big) \in \mathbb{B}_1^3\}$ where
$\theta \in [0, 2\pi], -ak_0 \leq t \leq ak_0$ and $k_0 > 0$ is the solution of $\cosh(k_0) = \frac{1}{k_0}$ e
$a = \frac{1}{\sqrt{\cosh^2(k_0)+k_0^2}}$

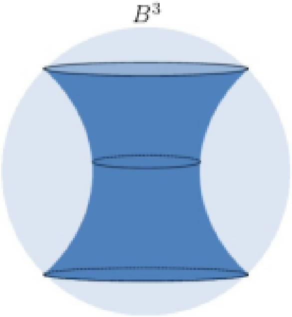

Figure: Image by Tayanara Santos.

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$$\frac{d}{dt}|_{t=0}|\Phi_t(M)| = -\int_\Sigma \langle X, H\rangle d\mu_\Sigma + \int_{\partial\Sigma} \langle X, \eta\rangle d\mu_{\partial\Sigma},$$

where

- η is the outward with unit conormal vector of $\partial\Sigma$.
- H is the mean curvature vector of Σ in M.
- $X = \dfrac{d\Phi}{dt}|_{t=0}$ is the variation field.
- $\dfrac{d}{dt}|_{t=0}|\Phi_t(M)| = 0 \Longleftrightarrow H = 0$ and $\Sigma \perp \partial M$ along $\partial\Sigma$.

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- $\dfrac{d}{dt}|_{t=0}|\Phi_t(M)| = 0 \iff H = 0$ and $\Sigma \perp \partial M$ along $\partial\Sigma$.
- The first variation formula shows that free boundary CMC surfaces are critical points of the area functional for volume preserving variations of Σ, whose $\partial\Sigma$ is free to move in ∂M.

Ros and Vergasta studied compact, stable, CMC hypersurfaces with free boundary in the ball $\mathbb{B} \subset \mathbb{R}^{n+1}$:

Theorem (Ros - Vergasta - 1995)

Let $\mathbb{B}^3 \subset \mathbb{R}^3$ be a closed ball. If $\Sigma^2 \subset \mathbb{B}^3$ is an immersed orientable compact stable CMC surface with free boundary, then $\partial\Sigma$ is embedded and the only possibilites are

1. *Σ is a totally geodesic disk.*
2. *Σ is a spherical cap;*
3. *Σ has genus 1 with at most two boundary components.*

Theorem (Nunes - 2017)

Let $\mathbb{B}^3 \subset \mathbb{R}^3$ *be a closed ball. If* $\Sigma^2 \subset \mathbb{B}^3$ *is an immersed orientable compact stable CMC surface with free boundary, then* Σ^2 *has genus zero.*

Theorem (Nunes - 2017)

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Remark:

As a consequence of Ros-Vergasta and Nunes, was complete classification of immersed compact stable CMC surfaces with free boundary in closed balls of $\mathbb{R}^3$.

Corollary (Nunes - 2017)

The totally umbilical disks are the only immersed orientable compact stable CMC surfaces with free boundary in a closed ball $\mathbb{B}^3 \subset \mathbb{R}^3$.

Existence and Classification

- Struwe - 1988, Jost and Grüter - 1986: showed the existence of properly embedded free boundary minimal disks inside strictly convex subsets of $\mathbb{R}^3$.

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- Folha, Pacard and Zolotareva - 2016: From a sufficiently large r there is a surface of genus 1 and r components on the boundary.
- Ketover - 2017: For any value of genus $g \geq 1$ there exists a minimal free boundary surface on $\mathbb{B}^3$ with genus g.

Geometric Analysis Gallery - Mario B. Schulz's homepage

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- Fraser and Schoen - 2015: generalized Nitsche's result to free boundary minimal disk in geodesic ball of arbitrary dimension in spaces forms, i.e., flat equatorial disk $\Sigma^2 \subset B^{n+1}$ is the only free boundary minimal disk in the balls of a space form.

GAP RESULTS

Theorem (Chern - do Carmo - Kobayashi - 1970; Lawson 1969)

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Let Σ^n be a closed minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$.Suposse that the second fundamental form A satisfies

$$|A|^2 \leq n.$$

Then,

- *either $|A|^2 = 0$ and Σ is an equator.*
- *or $|A|^2 = n$ and Σ is a Clifford hypersurface (product of two round spheres of appropriate radius and dimensions).*

Theorem (Ambrozio - Nunes - 2016)

Let Σ^2 be a compact free boundary minimal surface in $\mathbb{B}^3$. Assume that

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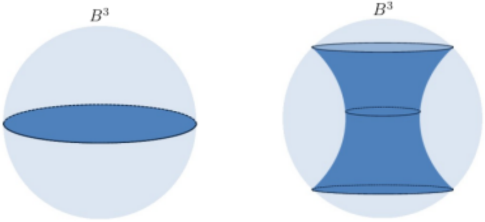
where $N(x)$ denotes an unit normal vector at the point $x \in \Sigma$ and A denotes the second fundamental form of Σ. Then

1. *either $|A|^2(x)\langle x, N\rangle^2 = 0$ and Σ is a flat equatorial disk;*
2. *$|A|^2(p)\langle p, N(p)\rangle^2 = 2$ at some point $p \in \Sigma$ and Σ is a critical catenoid.*



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- Σ is a critical catenoid by the solution to Björling problem.

Consider $X : [-t_0, t_0] \times [0, 2\pi] \to \mathbb{R}^3$ parametrization of a piece of catenoid contained in $\mathbb{B}^3$, meeting $\partial\mathbb{B}^3$ orthogonally

$$X(t, \theta) = (a_0 \cosh(t) \cos(\theta), a_0 \cosh(t) \sin(\theta), a_0 t),$$

the constant t_0 is the unique positive solution to the equation $t \sinh(t) = \cosh(t)$, while $a_0 = (t_0 \cosh(t_0))^{-1}$.

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Thus,

$$|A|^2 \langle x, N\rangle^2 = \frac{2}{\cosh^6(t)} (\cosh(t) - t \sinh(t))^2 \leq 2.$$

Conjecture (Fraser - Li - 2012)

The critical catenoid is the only embedded free boundary minimal annulus in B^3*, up to rigid motions.*

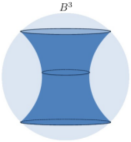

Figure: Image by Tayanara Santos.

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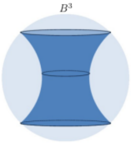

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Conjecture (Lawson-1970)

The Clifford torus is the only embedded minimal torus in $\mathbb{S}^3$, up to rigid motions.

It was proved by S. Brendle. S. Brendle, Embedded minimal tori in $\mathbb{S}^3$ and the Lawson conjecture, Acta Math. 211(2), 177-190 (2013).

Theorem (Barbosa - Cavalcante - Pereira - 2019)

Let Σ^2 be a compact free boundary CMC surface in B^3. Assume that for all points $x \in \Sigma$,

$$|\mathring{A}|^2 \langle x, N\rangle^2 \leq \frac{1}{2}(2 + H\langle x, N\rangle)^2, \tag{1}$$

where $\mathring{A} = II - \frac{H}{2} g_\Sigma$. Then

1. *either $|\mathring{A}|^2 \langle x, N\rangle^2 = 0$ and Σ is spherical cap.*
2. *or equality in (1) occours at some point and Σ is a part of Delaunay surface.*

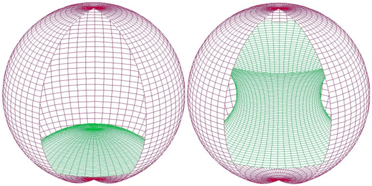

Figure: Image by Barbosa, Cavalcante and Pereira.

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- Check that there are some rotational CMC annulus in the unit Euclidean ball $\mathbb{B}^3$.

Andrade - Barbosa - Pereira results

- $\mathbb{B}_a^3 \subset \mathbb{R}^3$, $a < \infty$ or $a = \infty$,
- $u : \left[0, a^2\right) \to \mathbb{R}$ smooth,
- $h : \mathbb{B}_a^3 \to \mathbb{R}$ given by $h(x) = u(|x|^2)$,

Model Manifold

- $(\mathbb{B}_r^3, \bar{g})$.

Where

$$r < a$$
$$\bar{g} = e^{2h} \langle \, , \, \rangle$$

- If $u : [0, \infty) \to \mathbb{R}$ is given by $u(t) = 0$, then $\bar{g} = \langle \, , \, \rangle$.

$$(\mathbb{R}^3, \bar{g}) \sim \mathbb{R}^3$$

- If $u:[0,\infty)\to\mathbb{R}$ is given by $u(t)=0$, then $\bar{g}=\langle\,,\,\rangle$.

$$(\mathbb{R}^3,\bar{g})\sim\mathbb{R}^3$$

- If $u:[0,1)\to\mathbb{R}$ is given by $u(t)=\ln\left(\frac{2}{1-t}\right)$, then $\bar{g}=\frac{4}{(1-|x|^2)^2}\langle\,,\,\rangle$.

$$(\mathbb{B}^3_1,\bar{g})\sim\mathbb{H}^3$$

- If $u : [0, \infty) \to \mathbb{R}$ is given by $u(t) = \ln\left(\frac{2}{1+t}\right)$, then $\bar{g} = \frac{4}{(1+|x|^2)^2} \langle\, , \,\rangle$.

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- If $u : [0,\infty) \to \mathbb{R}$ is given by $u(t) = -\frac{t}{4}$, then $\bar{g} = e^{-\frac{|x|^2}{4}} \langle\, ,\, \rangle$.

$$(\mathbb{R}^3, \bar{g}) = \left(\mathbb{R}^3, e^{-\frac{|\vec{x}|^2}{4}} \langle , \rangle\right) \sim \mathbb{G}^3$$

- Let $\bar{\nabla}$ and ∇ be the Riemannian conection of $(\mathbb{B}_r^3, \bar{g})$ e $(\mathbb{B}_r^3, \langle\,,\,\rangle)$ respectively.
- $\vec{x}$ position field in $\mathbb{B}_r^3$

$(\mathbb{B}_r^3, \langle\,,\,\rangle)$

$$\nabla_Y \vec{x} = Y, \quad \mathcal{L}_{\vec{x}} \langle\,,\,\rangle = 2\langle\,,\,\rangle$$

$(\mathbb{B}_r^3, \bar{g})$

$$\bar{\nabla}_Y \vec{x} = \sigma Y, \quad \mathcal{L}_{\vec{x}} \bar{g} = 2\sigma \bar{g}$$

where

$$\sigma(x) = 1 + 2u'(|x|^2)|x|^2$$

Theorem (A., Barbosa, Pereira)

Let Σ^2 be a compact free boundary CMC surface in $(\mathbb{B}^3_r, \bar{g})$. Suppose that for all points $x \in \Sigma$,

$$\begin{cases} \dfrac{|\mathring{A}|^2}{\sigma^2}\bar{g}(\vec{x},N)^2 & \leq & \dfrac{1}{2}\left(2+\dfrac{H}{\sigma}\bar{g}(\vec{x},N)\right)^2 \\ 0 & \leq & 2+\dfrac{H}{\sigma}\bar{g}(\vec{x},N). \end{cases}$$

Then, one of the following situations occurs,

- i) *either Σ is diffeomorphic to a disk,*
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- The set

$$\mathcal{C} = \{p \in \Sigma; \Psi(p) = min_\Sigma \Psi(x)\},$$

is totally convex in Σ.

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- V is tangent to Σ.
- Since the vector field V on $\mathbb{R}^3$ is induced by rotations around the axis that is orthogonal to the plane π, and passes through the center of the great circle which contains γ, we see that Σ is rotationally symmetric.

Corollary

Let Σ be a compact free boundary minimal surface in $(\mathbb{B}^3_r, \bar{g})$. Suppose that for all points $x \in \Sigma$,

$$\frac{|A|^2}{\sigma^2} \bar{g}(\vec{x}, N)^2 \quad \leq \quad 2.$$

Then, one of the following situations occurs,

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Corollary

Let Σ be a compact free boundary CMC surface in $(\mathbb{B}^3_r, \bar{g})$. If

$$\frac{|\mathring{A}|^2}{\sigma^2}\bar{g}(\vec{x}, N)^2 < \frac{1}{2}\left(2 + \frac{H}{\sigma}\bar{g}(\vec{x}, N)\right)^2,$$

then Σ is diffeomorphic to a disk $\mathbb{D}^2$.

Theorem (A., Barbosa, Pereira)

There is $r > 0$ and a minimal surface $\Sigma \subset (\mathbb{B}^3_r, e^{-\frac{|\vec{x}|^2}{4}} \langle , \rangle)$ with strictly convex boundary $\partial\Sigma \subset \partial\mathbb{B}^3_r$ where the condition

$$\frac{1}{\sigma^2} |A|^2 \, \bar{g}(\vec{x}, \bar{N})^2 \leq 2$$

is satisfied for all point p in Σ.

Remark:

- *Li and Xiong (2017) proved the analogous results for FBMS in a geodesic ball of* $\mathbb{S}^3_+$ *and* $\mathbb{H}^3$.

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We recall that the traceless second fundamental form is defined as

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Theorem (Cavalcante, Mendes, Vitório - 2018)

Let Σ^2 be a free boundary compact orientable surface immersed in $\mathbb{B}^{2+k}$, for any positive integer k. If $|\Phi|^2 \leq 2$, then Σ^2 is topologically a disk.

Proof.

Use the blackboard. □

Corollary (Cavalcante, Mendes, Vitório - 2018)

Let Σ^2 be a free boundary compact orientable surface minimally immersed in $\mathbb{B}^{2+k}$, for any positive integer k. If $|A|^2 \leq 4$, then Σ^2 is the flat equatorial disk.

Proof.

Use the blackboard. □

Remark:

- *Li and Xiong (2017) proved the analogous results for FBMS in a geodesic ball of* $\mathbb{S}^3_+$ *and* $\mathbb{H}^3$.
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- *Cavalcante, Mendes and Vitório (2018)* $\Sigma^2 \subset \mathbb{B}^{2+k}, k \geq 1$.
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- *This same gap condition was used by Meeks, Pérez and Ros (2008) to characterize the plane and the catenoid in* $\mathbb{R}^3$ *among properly embedded mininmal surfaces without boundary.*

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1. Lecture 1
   - (Brief) Motivation to study Differential Geometry. (done)
2. Lecture 2
   - Free boundary minimal or CMC surfaces. Gap results. (done)
3. Lecture 3
   - Free boundary CMC (hyper)surface in the ball. Stability.
4. Lecture 4
   - Some characterization of the critical catenoid.

References:

AMBROZIO, LUCAS; NUNES, IVALDO, *A gap theorem for free boundary minimal surfaces in the three-ball.* arXiv preprint arXiv:1608.05689 (2016).

ANDRADE, MARIA; BARBOSA, EZEQUIEL; PEREIRA, EDNO, *Gap Results for Free Boundary CMC Surfaces in Radially Symmetric Conformally Euclidean Three-Balls*, The Journal of Geometric Analysis, v. 31, p. 8013-8035, 2021.

ROS, ANTONIO; VERGASTA, ENALDO, *Stability for hypersurfaces of constant mean curvature with free boundary*, Geometriae Dedicata, 56, (1995), 19–33.

BARBOSA, EZEQUIEL; CAVALCANTE, MARCOS P.; PEREIRA, EDNO, *Gap results for free boundary CMC surfaces in the Euclidean three-ball* arXiv preprint arXiv:1908.09952 (2019).

NUNES, IVALDO, *On stable constant mean curvature surfaces with free boundary* Mathematische Zeitschrift 287.1 (2017): 473-479.

Many others ...

Thank you for your attention!
See you tomorrow!