FREE BOUNDARY CONSTANT MEAN CURVATURE HYPERSURFACES

Maria Andrade
Universidade Federal de Sergipe -Brazil
Lecture 3

CIMPA/ICTP RESEARCH IN PAIRS

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- This is a mini-course at master/beginning of PhD level.
  1. Some classic results.
  2. Free boundary CMC (minimal) (hyper)surfaces in the ball.
  3. Gaps results.
  4. Stability.
  5. Index.
  6. The Steklov Eigenvalue Problem.
  7. Some characterization of the critical catenoid.
  8. Some open problems.

1. Lecture 1
   - (Brief) Motivation to study Differential Geometry. (done)
2. Lecture 2
   - Free boundary minimal or CMC (hyper)surfaces. Gap results. (done)
3. Lecture 3
   - Free boundary CMC (hyper)surface in the ball. Stability.
4. Lecture 4
   - Some characterization of the critical catenoid. Index.

Lecture 3

A proper minimal hypersuface Σ^{n-1} of the unit ball B^n which is orthogonal to the sphere at the boundary is called a free boundary minimal hypersurface.

- $\Sigma \subset \mathbb{R}^n$ minimal $\Longleftrightarrow \Delta_\Sigma x_i = 0$, for $i = 1, ..., n$
- Σ meets $\partial\mathbb{B}^n$ $\Longleftrightarrow$ $\frac{\partial x_i}{\partial \eta} = x_i$, for $i = 1, ..., n$

$$\begin{cases} \Delta_\Sigma x_i &= 0, \text{ on } \Sigma \\ \frac{\partial x_i}{\partial \eta} &= 1 x_i, \text{ on } \partial\Sigma \end{cases}$$

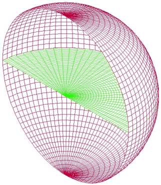

Figure: Image by Barbosa - Cavalcante - Pereira

Consider Σ^k a compact $k-$dimensional immersed submanifold an n-dimensional Riemannian manifold M with boundary $\partial\Sigma \subset \partial M$. Let

$$\Phi_t : \Sigma \to M$$

be an one-parameter family of immersions with $\Phi_t(\partial\Sigma) \subset \partial M,\ t \in (-\epsilon, \epsilon)$, with F_0 given by the inclusion $\Sigma \hookrightarrow M$. The first variation of volume is:

$$\frac{d}{dt}|_{t=0}|\Phi_t(M)| = -\int_\Sigma \langle X, H\rangle d\mu_\Sigma + \int_{\partial\Sigma} \langle X, \eta\rangle d\mu_{\partial\Sigma},$$

where

- η is the outward with unit conormal vector of $\partial\Sigma$.
- H is the mean curvature vector of Σ in M.
- $X = \dfrac{d\Phi}{dt}|_{t=0}$ is the variation field.
- $\dfrac{d}{dt}|_{t=0}|\Phi_t(M)| = 0 \Longleftrightarrow H = 0$ and $\Sigma \perp \partial M$ along $\partial\Sigma$.
- The first variation formula shows that free boundary CMC surfaces are critical points of the area functional for volume preserving variations of Σ, whose $\partial\Sigma$ is free to move in ∂M.

Theorem

[1] *Let $x : \Sigma^2 \to \mathbb{B}^3$ be an immersed surface with $x(\partial\Sigma) \subset \mathbb{S}^2$. Then the following statements are equivalent:*

1. *Σ is a free boundary minimal surface (i.e. $H = 0$ and $\eta \perp \mathbb{S}^2$.)*
2. *Σ is a critical point of the area functional among surface in $\mathbb{B}^3$ with boundary lying on S^2.*
3. *The coordinate functions x_i of $\mathbb{R}^3$ restrict to harmonic functions on Σ and they satisfies the boundary condition $\frac{\partial x_i}{\partial \eta} = x_i$.*

$$\begin{cases} \Delta_\Sigma x_i & = & 0, \text{ on } \Sigma \\ \frac{\partial x_i}{\partial \eta} & = & x_i, \text{ on } \partial\Sigma \end{cases}$$

[1] Li, M.; FREE BOUNDARY MINIMAL SURFACES IN THE UNIT BALL : RECENT ADVANCES AND OPEN QUESTIONS, Arxiv, 2019.

Suppose Σ^{n-1} is a two-sided free boundary minimal hypersurfaces in M. Then, the second variation of Σ, for normal variations $X = \varphi N$, where N is an unit normal field to Σ and $\varphi \in C^\infty(\Sigma)$ is given by

$$\frac{d^2}{dt^2}|_{t=0}|\Phi_t(\Sigma)| = Q(\varphi, \varphi),$$

$$\begin{aligned} Q(\varphi,\varphi) &= \int_\Sigma (|\nabla\varphi|^2 - Ric_M(N,N)\varphi^2 - |A|^2\varphi^2)d\mu_\Sigma + \int_{\partial\Sigma} \varphi^2 \langle \nabla_N N, \eta\rangle d\mu_{\partial\Sigma}. \\ &= -\int_\Sigma \varphi L\varphi d\mu_\Sigma + \int_{\partial\Sigma} \left(\frac{\partial\varphi}{\partial\eta} - \varphi II^{\partial M}(N,N)\right)\varphi d\mu_{\partial\Sigma}, \end{aligned}$$

where $L = \Delta + Ric(N,N) + |A|^2$ is call the Jacobi operator.

Consider $\Sigma \subset \Omega \subset \mathbb{R}^3$. We say that Σ is stable if it has nonnegative second variation of area for all volume variations. This means:

1. The mean curvature H of Σ is constant;
2. Σ meets $\partial\Omega$ orthogonally along $\partial\Sigma$;
3. (Stability)

$$Q(\varphi, \varphi) = \int_\Sigma (|\nabla \varphi|^2 - |A|^2 \varphi^2) da - \int_{\partial\Sigma} II^{\partial\Omega}(N, N) \varphi^2 dl \geq 0,$$

$$\varphi \in H^1(\Sigma); \int_\Sigma \varphi da = 0.$$

Ros and Vergasta studied compact, stable, CMC hypersurfaces with free boundary in the ball $\mathbb{B} \subset \mathbb{R}^{n+1}$:

Theorem (Ros - Vergasta - 1995)

Let $\mathbb{B}^3 \subset \mathbb{R}^3$ be a closed ball. If $\Sigma^2 \subset \mathbb{B}^3$ is an immersed orientable compact stable CMC surface with free boundary, then $\partial\Sigma$ is embedded and the only possibilites are

1. *Σ is a totally goedesic disk.*
2. *Σ is a spherical cap;*
3. *Σ has genus 1 with at most two boundary components.*

Theorem (Nunes - 2017)

Let $\mathbb{B}^3 \subset \mathbb{R}^3$ be a closed ball. If $\Sigma^2 \subset \mathbb{B}^3$ is an immersed orientable compact stable CMC surface with free boundary, then Σ^2 has genus zero.

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Remark:

As a consequence of Ros-Vergasta and Nunes, was complete classification of immersed compact stable CMC surfaces with free boundary in closed balls of $\mathbb{R}^3$.

Corollary (Nunes - 2017)

The totally umbilical disks are the only immersed orientable compact stable CMC surfaces with free boundary in a closed ball $\mathbb{B}^3 \subset \mathbb{R}^3$.

Theorem (Nunes - 2017)

Let $\Omega \subset \mathbb{R}^3$ be a smooth compact convex domain. Suppose that the second fundamental form $II^{\partial\Omega}$ of Ω satisfies the pinching condition

$$kh \leq II^{\partial\Omega} \leq (3/2)hk,$$

for some constant $k > 0$, where h denotes the induced metric on $\partial\Omega$. If $\Sigma \subset \Omega$ is an immersed orientable compact stable CMC surface with free boundary, then Σ has genus zero and $\partial\Sigma$ at most two connected.

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Proposition

Let $\Omega \subset \mathbb{R}^3$ be a compact convex domain. If $\Sigma \subset \Omega$ is an immersed stable CMC surface with free boundary. Then,

$$Q^0(\varphi,\varphi) = \int_\Sigma (|\nabla\varphi|^2 - |A|^2\varphi^2)da \geq 0, \ \forall\varphi; \ \varphi = 0 \text{ on } \partial\Sigma.$$

Proof.

Use the blackboard. □

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Theorem (Nunes - 2017)

Let $\Omega \subset \mathbb{R}^3$ be a smooth compact convex domain. Suppose that the second fundamental form $II^{\partial\Omega}$ of Ω satisfies the pinching condition

$$kh \leq II^{\partial\Omega} \leq (3/2)hk, \tag{1}$$

for some constant $k > 0$, where h denotes the induced metric on $\partial\Omega$. If $\Sigma \subset \Omega$ is an immersed orientable compact stable CMC surface with free boundary, then Σ has genus zero and $\partial\Sigma$ at most two connected.

Proof.

Blackboard. $\square$

Theorem (Ros-Vergasta - 1995)

Let $\mathbb{B} \subset \mathbb{R}^{n+1}, n \geq 2,$ *be a closed ball. Let* $\Sigma \subset \mathbb{B}$ *be a CMC free-boundary stable hypersurface with embedded boundary in* $\mathbb{B}$. *If* $L \geq nA$, *then* Σ *is totally geodesic or starshaped with respect to the center of the ball.*

Theorem (Barbosa - 2018)

Let $\mathbb{B} \subset \mathbb{R}^{n+1}, n \geq 2,$ *be an unit closed ball. If* $\Sigma \subset \mathbb{B}$ *is a CMC free-boundary stable hypersurface in* $\mathbb{B}$, *then*

$$\frac{nH^2}{2} \int_{\Sigma} (1 - |x|^2) dvol_{\Sigma} + nA \leq L \leq nA(1 + H), \tag{2}$$

where the orientation of Σ *is in a such way* $H \geq 0$.

Theorem (Ros-Vergasta - 1995)

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$$\frac{nH^2}{2} \int_{\Sigma} (1 - |x|^2) dvol_{\Sigma} + nA \leq L \leq nA(1 + H), \qquad (2)$$

where the orientation of Σ *is in a such way* $H \geq 0$*.The left side of* (2) *is an equality if, and only if,* Σ *is a totally geodesic disk or a spherical cap. In particular, if* $\partial\Sigma$ *is embedded, then* Σ *is either totally geodesic or starshaped with respect to center of the ball.*

Lemma (Nunes Stability Type Lemma)

Let $\mathbb{B}$ be a compact convex domain in $\mathbb{R}^{n+1}$ and Σ an immersed stable hypersurface CMC with free boundary in $\mathbb{B}$. If $f \in C^\infty(\Sigma)$ is such that $f(x) = 0$ for every $x \in \partial\Sigma$, then

$$Q(f,f) = \int_\Sigma (|\nabla f|^2 - |A_\Sigma|^2 f^2) dvol_\Sigma \geq \frac{1}{n+1} \left(\frac{\int_\Sigma f dvol_\Sigma}{A} \right)^2 \int_{\partial\Sigma} II(N,N) ds.$$

and equality holds for some non null function f, if and only if, Σ is totally geodesic.

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and equality holds for some non null function f, if and only if, Σ is totally geodesic.In particular, if $f \in C^\infty(\Sigma)$ is such that $f(x) = 0$ for all $x \in \partial\Sigma$, then

$$Q(f,f) = \int_\Sigma (|\nabla f|^2 - |A_\Sigma|^2 f^2) dvol_\Sigma \geq 0.$$

Theorem (Barbosa - 2018)

Let $\mathbb{B} \subset \mathbb{R}^{n+1}, n \geq 2,$ *be an unit closed ball. If* $\Sigma \subset \mathbb{B}$ *is a CMC free-boundary stable hypersurface in* $\mathbb{B}$, *then*

$$\frac{nH^2}{2} \int_\Sigma (1 - |x|^2) dvol_\Sigma + nA \leq L \leq nA(1 + H), \tag{3}$$

where the orientation of Σ *is in a such way* $H \geq 0$. *The left side of* (3) *is an equality if, and only if,* Σ *is a totally geodesic disk or a spherical cap. In particular, if* $\partial\Sigma$ *is embedded, then* Σ *is either totally geodesic or starshaped with respect to center of the ball.*

Proof.

Blackboard □

Theorem (Wang-Xia - 2019)

Any stable immersed CMC hypersurfaces with free boundary in an Euclidean ball is eitheir a totally geodesic ball or a spherical cap.

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References:

- BARBOSA, EZEQUIEL, *On CMC free-boundary stable hypersurfaces in a Euclidean ball*, Mathematische Annalen, 372, (2018), 179–187.
- ROS, ANTONIO; VERGASTA, ENALDO, *Stability for hypersurfaces of constant mean curvature with free boundary*, Geometriae Dedicata, 56, (1995), 19–33.
- NUNES, IVALDO, *On stable constant mean curvature surfaces with free boundary* Mathematische Zeitschrift 287.1 (2017): 473-479.
- Many others ...

Thank you for your attention!
See you tomorrow!