FREE BOUNDARY CONSTANT MEAN CURVATURE HYPERSURFACES

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Lecture 4

CIMPA/ICTP RESEARCH IN PAIRS

Supported also by CNPq Grant 403349/2021-4

June 15, 2023

- This is a mini-course at master/beginning of PhD level.
  1. *Free boundary CMC (minimal) (hyper)surfaces in the ball.*
  2. *Gaps results.*
  3. *Stability.*
  4. *Index.*
  5. *The Steklov Eigenvalue Problem.*
  6. *Some characterization of the critical catenoid.*
  7. *Some open problems.*

1. Lecture 1
    - (Brief) Motivation to study Differential Geometry. (done)
2. Lecture 2
    - Free boundary minimal or CMC (hyper)surfaces. Gap results. (done)
3. Lecture 3
    - Free boundary CMC (hyper)surface in the ball. Stability. (done)
4. Lecture 4
    - Some characterization of the critical catenoid. Index.

Lecture 4

A proper minimal submanifold Σ of the unit ball B^n which is orthogonal to the sphere at the boundary is called a free boundary submanifold.

- $\Sigma \subset \mathbb{R}^n$ minimal $\Longleftrightarrow \Delta_\Sigma x_i = 0$, for $i = 1, ..., n$
- Σ meets $\partial\mathbb{B}^n \Longleftrightarrow \frac{\partial x_i}{\partial \nu} = x_i$, for $i = 1, ..., n$

$$\begin{cases} \Delta_\Sigma x_i & = & 0, \text{ on } M \\ \frac{\partial x_i}{\partial \nu} & = & 1 x_i, \text{ on } \partial M \end{cases}$$

Theorem (Fraser-Schoen)

FBMH are characterized by condition that the coordinate functions are Steklov eigenvalue 1, that is, $\Delta x_i = 0$ *and* $\nabla_\eta x_i = x_i,\ i = \{1, \cdots, n\}$

Some ideia.

Let $x_i : \Sigma \to \mathbb{R}\ i = 1, \cdots, n$ be the coordiante functions.

$$p \in \Sigma \mapsto x_i = \langle p, e_i \rangle.$$

- $\nu = x$(position vector on $\partial\Sigma$) $\Rightarrow x_i = \nu_i = \langle \nu, e_i \rangle = \frac{\partial}{\partial \nu} x_i,\ \forall i = 1, \cdots, n.$
- $\nabla x_i = e_i - e_i^\top$
- $\Delta x_i = H_i.$

$\square$

(M^n, g) be a compact n-dimensional Riemannian. A function u on M is a Steklov eigenfunction with eigenvalue σ if

$$\begin{cases} \Delta_g u = 0, \text{ on } M. \\ \frac{\partial u}{\partial \nu} = \sigma u, \text{ on } \partial M. \end{cases}$$

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Steklov eigenvalues are eigenvalues of the Dirichlet-to-Neumann operator:

$$L : C^\infty(\partial M) \to C^\infty(\partial M)$$

given by

$$Lu = \frac{\partial \hat{u}}{\partial \nu},$$

where $\hat{u}$ is the harmonic extension of u to M

$$\begin{cases} \Delta_g \hat{u} &= 0, \text{ on } M. \\ \hat{u} &= u, \text{ on } \partial M. \end{cases}$$

L is a self-adjoint operator which is non-negative definitive.

$\sigma_0, \sigma_1, \cdots, \sigma_n$ eigenvalues and $u_0, u_1, \cdots, u_n$ eigenfunctions.

$$\begin{cases} \Delta_g u_i &= 0, \text{ on } M \\ \frac{\partial u_i}{\partial \nu} &= \sigma_i u_i, \text{ on } \partial M \end{cases}$$

$\sigma_0 = 0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_k \leq \cdots \to \infty$

$$\sigma_1 = \inf_{u \in C^1(\partial M); \int_{\partial M} u = 0} \frac{\int_M |\nabla \hat{u}|^2 da}{\int_{\partial M} u^2 ds}$$

Question

[1] *How big can the first eigenvalue be?*

[1] Fraser, Ailana, et al. Geometric Analysis. Springer International Publishing, 2020.

Theorem (Weinstock-1954)

Let Ω be a simply connected plane domain. Then

$$\sigma_1 L(\partial\Omega) \leq 2\pi,$$

where $L(\partial\Omega)$ is the length of $\partial\Omega$. Equality is achieved if and only if Ω is a disk.

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- *The disk uniquely maximizes σ_1 among simply connected domains with the same boundary length.*
- *After Weinstock proved this result for surface Σ simply connected with boundary.*

Theorem (Fraser - Schoen - 2011)

Let (Σ, g) be a compact Riemannian surface with genus γ and k boundary components. Then,

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- *The proof uses a result of Ahlfors with an improvement by Gabard to construct proper holomorphic maps from Σ to the unit disk controlled degree. After is used the ideia used by Szego and Weinstock to use automorphism of the disk to balance the map and constructed test functions.*

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Proof.

Use the black board. □

Theorem (Fraser-Schoen - 2011)

Let Σ be a compact surface of genus 0 with k boundary components, $k \geq 2$. Let σ_1 be the first non-zero eingenvalue of the Dirichlet-to-Neumann operator of Σ with metric g. Then,

$$\sigma_1 L(\partial M) < 2k\pi.$$

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This shows that bound given before is not sharp.

Theorem (Fraser-Schoen-2011)

Let $\Sigma^2 \subset B^n$ minimal, $\partial\Sigma \neq \emptyset$, $\partial\Sigma \subset \partial\mathbb{B}^n$ and meeting $\partial\mathbb{B}^n$ orthogonally along $\partial\Sigma$. Then

$$2A(\Sigma) = L(\partial\Sigma) \geq 2\pi.$$

Corollary

The sharp isoperimetric inequality holds for free boundary minimal surfaces in the ball

$$A(\Sigma) \leq \frac{L^2(\partial\Sigma)}{4\pi}.$$

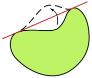

Figure: Image by Wikipedia

Proof.

Use the blackboard. □

Theorem (Fraser - Schoen - 2016)

For every $k \geq 1$ there is an embedded free boundary minimal surface in $\mathbb{B}^3$ of genus 0 with k boundary components. Moreover, these surfaces are embedded by first Steklov eigenvalue.

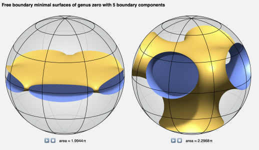

Figure: *Image by Mario B. Schulz.*

Remark:

- *Previously, the only know free boundary minimal surfaces in $\mathbb{B}^3$ were the equatorial plane disk and the critical catenoid.*

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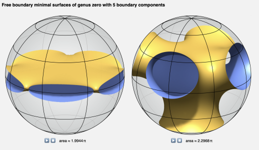

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Remark:

- *Previously, the only know free boundary minimal surfaces in* $\mathbb{B}^3$ *were the equatorial plane disk and the critical catenoid.*
- *This theorem and the connection between free boundary minimal surfaces in a ball and the Steklov eigenvalue problem have raised the interest of finding more examples of properly embedded minimal surfaces in the unit ball.*

Question

Which compact orientable surfaces with boundary can be realized as properly embedded free boundary minimal surfaces in $\mathbb{B}^3$*?*

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Which compact orientable surfaces with boundary can be realized as properly embedded free boundary minimal surfaces in $\mathbb{B}^3$?

- Fraser-Schoen - 2016: $\gamma = 0,\ k \geq 1.$
- Folha - Pacard - Zolotareva -2017: $\gamma = 1\ k \gg .$
- Kapouleas - Li: $\gamma \gg,\ k = 3.$
- Kapouleas - Wiygul: $\gamma \geq 0,\ k = 1.$

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Given a compact orientable surface with boundary in how many ways can one realize it is a properly embedded free boundary minimal surfce in unit ball $\mathbb{B}^n$?

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A classical results of Nitsche shows that the only free boundary minimal disk in the ball $\mathbb{B}^3$ is the equatorial plane disk.

Theorem (Nitsche-1985)

The equatorial disk is the only (up to rigid motions of $\mathbb{B}^3$) immersed free boundary minimal disk in $\mathbb{B}^3$.

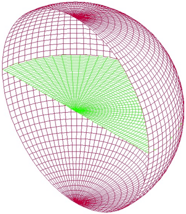

Figure: Image by Barbosa - Cavalcante - Pereira

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- *Fraser-Schoen proved any free boundary minimal disk on $\mathbb{B}^n$ is an equatorial plane disk.*
- *This result is surprising by analogy with the case of minimal S^2 in $\mathbb{S}^3$ is totally geodesics, however there are many minimal immersions of S^2 in $\mathbb{S}^n$ for $n \geq 4$ that are not totally geodesic.*
- *The critical catenoid is expected to be only embedded free boundary annulus in $\mathbb{B}^3$.*

Conjecture (Fraser-Li - 2012)

The critical catenoid is the unique embedded free boundary minimal surface in $\mathbb{B}^3$ *that is homemorphic to an annulus.*

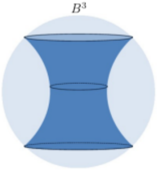
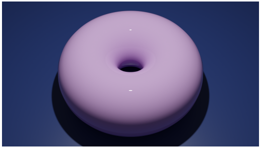


Figure: Image by Tayanara Santos

Analogy of the Lawson conjecture on the uniqueness of the Clifford torus in $\mathbb{S}^3$ (proved by S. Brendle).



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Assume that Σ is a free boundary minimal annulus in $\mathbb{B}^n$ such that the coordinate functions are first Steklov eigenvalues. Then $n = 3$ and Σ is the critical catenoid.

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- *This result characterizes the critical catenoid as the only free boundary minimal annulus in $\mathbb{B}^n$ such that coordinate functions are first Steklov eigenvalues, i.e., $\sigma_1(\Sigma) = 1$.*

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- *McGrath used this result to prove that the catenoid is the only embedded free boundary annulus in $\mathbb{B}^3$ that is invariant under reflection through the coordinates planes. This provideng further evidence the conjecture about critical catenoid.*

Conjecture (Frazer - Li - 2012)

Let Σ *be a compact properly embedded free boundary minimal hypersurface in* $\mathbb{B}^n$*. Then* $\sigma_1(\Sigma) = 1$*.*

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- *Fraser - Li proved that $\sigma_1(\Sigma) \geq 1/2$.*
- *After Batista - Cunha showed that $\sigma_1(\Sigma) > 1/2$.*

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Remark:

The result by Ros gives evidence to this conjecture.

$$
\begin{aligned}
\varphi : \Sigma \to \mathbb{R};\ \varphi(x) = \langle x, \nu \rangle \quad &\Rightarrow \quad \Delta_\Sigma \varphi = 2\varphi \\
{}''Courant'' \quad &\Rightarrow \quad \#\{\{\varphi > 0\} \cup \{\varphi < 0\}\} \leq 2 \\
&\Rightarrow \quad 2 - piece.
\end{aligned}
$$

Theorem (Lima, Menezes, 2019)

Let $\Sigma \subset \mathbb{B}^3$ *be a compact embedded free boundary minimal surface. Then, for any equatorial disk* D*, either* $M = D$*, or* D *divides* M *in exactly two-components.*

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Conjecture (Fraser-Li, 2012)

Let $M \subset \mathbb{B}^3$ be a compact embedded free boundary minimal surface. Then the first non-zero Steklov eigenvalue of M is equal to 1.

Remark:

The result by Lima - Menezes gives evidence to this conjecture.

$$\begin{aligned} \varphi : M \to \mathbb{R};\ \varphi(x) = \langle x, \nu \rangle \quad &\Rightarrow \quad \Delta_M \varphi = 0 \text{ and } \frac{\partial \varphi}{\partial \nu} = \varphi \\ &\Rightarrow \quad 2 - piece. \end{aligned}$$

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Corollary (Kapouleas - Li)

The only embedded free boundary minimal surface in $\mathbb{B}^3$ *with at least one rotationally invariant (about the* $z-$*axis) boundary component on* $\mathbb{B}^3$ *are the equatorial disk* $\mathbb{D}$ *and the critical catenoid* $\mathbb{K}$*.*

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This Theorem shows that the embeddedness assumption in the catenoid critical conjecture cannot removed.

Consider Σ^k a compact $k-$dimensional immersed submanifold an n-diemnsional Riemannian manifold M with boundary $\partial\Sigma \subset \partial M$. Let

$$\Phi_t : \Sigma \to M$$

is an one-parameter family of immersions with $\Phi_t(\partial\Sigma) \subset \partial M,\ t \in (-\epsilon, \epsilon)$, with F_0 given by the inclusion $\Sigma \hookrightarrow M$. The first variation of volume is:

$$\frac{d}{dt}|_{t=0}|\Phi_t(M)| = -\int_\Sigma \langle X, H\rangle d\mu_\Sigma + \int_{\partial\Sigma} \langle X, \eta\rangle d\mu_{\partial\Sigma},$$

where

- η is the outward with unit conormal vector of $\partial\Sigma$.
- H is the mean curvature vector of Σ in M.
- $X = \dfrac{d\Phi}{dt}|_{t=0}$ is the variation field.
- $\dfrac{d}{dt}|_{t=0}|\Phi_t(M)| = 0 \Longleftrightarrow H = 0$ and $\Sigma \perp \partial M$ along $\partial\Sigma$.

Suppose Σ^{n-1} is a two-sided free boundary minimal hypersurfaces in M. Then, the second variation of Σ, for normal variations $X = \varphi N$, where N is an unit normal field to Σ and $\varphi \in C^\infty(\Sigma)$ is given by

$$\frac{d^2}{dt^2}|_{t=0}|\Phi_t(\Sigma)| = Q(\varphi, \varphi),$$

$$\begin{aligned} Q(\varphi,\varphi) &= \int_\Sigma (|\nabla\varphi|^2 - Ric_M(N,N)\varphi^2 - |A|^2\varphi^2) d\mu_\Sigma + \int_{\partial\Sigma} \varphi^2 \langle \nabla_N N, \eta\rangle d\mu_{\partial\Sigma}. \\ &= -\int_\Sigma \varphi L\varphi d\mu_\Sigma + \int_{\partial\Sigma} \left(\frac{\partial\varphi}{\partial\eta} - \varphi h^{\partial M}(N,N)\right)\varphi d\mu_{\partial\Sigma}, \end{aligned}$$

where $L = \Delta + Ric(N,N) + |A|^2$ is call the Jacobi operator.

Remark:

- *Any free boundary minimal hypersurface* Σ *in a manifold with nonnegative Ricci curvature and convex boundary* ∂M *is unstable, because, if we take* $\varphi = 1$, *then*

$$Q(1,1) = -\int_{\Sigma}(Ric(N,N) + |A|^2))d\mu_{\Sigma} - \int_{\partial\Sigma} h^{\partial M}(N,N)d\mu_{\partial\Sigma} < 0.$$

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- *Consider* $M = \mathbb{B}^n$ *and* $\Sigma^{n-1} \subset \mathbb{B}^n$ *be a free boundary hypersurface. Then,*

$$\begin{aligned} Q(\varphi,\varphi) &= \int_{\Sigma}\left(|\nabla\varphi|^2 - |A|^2\varphi^2\right)d\mu_{\Sigma} - \int_{\partial\Sigma}\varphi^2 d\mu_{\partial\Sigma} \\ &= -\int_{\Sigma}\varphi L\varphi d\mu_{\Sigma} + \int_{\partial\Sigma}\left(\frac{\partial\varphi}{\partial\eta} - \varphi\right)\varphi d\mu_{\partial\Sigma}, \end{aligned}$$

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Definition

The *index* *of Σ is the maximal dimension a subespace of $C^\infty(\Sigma)$ on which the index form Q is negative definite, or equivalently, the number of negative eigenvalues of the Jacobi operator with Robin boundary condition:*

$$\begin{cases} L\varphi + \lambda\varphi &= 0, \text{ on } M \\ \frac{\partial\varphi}{\partial\nu} &= \varphi, \text{ on } \partial M \end{cases}$$

- If Σ is an equatorial hyperplane in $\mathbb{B}^n$, then $\mathrm{ind}(\Sigma) = 1$.
- If Σ is a free boundary minimal submanifold in the ball $\mathbb{B}^n$, then the coordinate functions $x_1, \cdots, x_n$ are Steklov eigenfunctions with eigenvalue 1.

- If Σ is not an equatorial hyperplane and φ is a Steklov eigenfunction of Σ with eigenvalue $\sigma_1 \leq 1$,

$$\begin{cases} \Delta_g \varphi &= 0, \text{ on } M \\ \frac{\partial \varphi}{\partial \nu} &= \sigma_1 \varphi, \text{ on } \partial M. \end{cases}$$

Then

$$\begin{aligned} Q(\varphi, \varphi) &= -\int_\Sigma (\Delta \varphi + |A|^2 \varphi)\varphi d\mu_\Sigma + \int_{\partial\Sigma} \left(\frac{\partial \varphi}{\partial \eta} - \varphi \right) \varphi d\mu_{\partial\Sigma}, \\ &= -\int_\Sigma |A|^2 \varphi^2 d\mu_\Sigma + (\sigma_1 - 1) \int_{\partial\Sigma} \varphi^2 d\mu_{\partial\Sigma}, \\ &< 0. \end{aligned}$$

- If Σ^{n-1} is a free-boundary minimal hypersurface in the ball $\mathbb{B}^n$ that is not an equatorial hyperplane, then

$$\text{ind}(\Sigma) \geq n+1.$$

Proof.

Proof in the blackboard. $\square$

Remark:

- *In particular, any free boundary minimal surface in the* B^3, *that is not an equatorial plane disk has index at least 4.*

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- *Devyver, Smith and Zhou, and Tran independently proved that the critical catenoid has index 4.*
- *The Clifford torus is caracterized as the unique closed minimal surface* $\mathbb{S}^3$ *of index 5 (Urbano - 1990). This index characterization plays a key role in the recent celebrated proof of the Willmore conjecture by F. Marques and A. Neves.*

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- *In particular, any free boundary minimal surface in the B^3, that is not an equatorial plane disk has index at least 4.*
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- *It is conjectured that te critical catenoid is the unique free boundary minimal surface in $\mathbb{B}^3$ of index 4.*

Conjecture

If Σ is a free boundary minimal surface in $\mathbb{B}^3$ of index 4, then Σ is the critical catenoid.

Theorem (Fraser- Schoen)

Suppose $\Sigma^2 \subset \mathbb{B}^3$ is a properly immersed FBMS with Morse index 4. Then the first Steklov eigenvalue is 1.

Proof.

Blackboard. $\square$

- In higher dimensions, the Clifford hypersurfaces in $\mathbb{S}^n$ have index $n+2$ and are conjectured to be only closed minimal hypersurfaces in $\mathbb{S}^n$ of index $n+2$. In contrast Smith et al. recently proved that the higher dimensional free boundary minimal catenoids Σ^{n-1} in $\mathbb{B}^n$ have suprisingly high index.

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- En general, if Σ^{n-1} is a free boundary minimal hypersurface in $\mathbb{B}^n$ of index n+1, then the first Steklov eigenvalue $\sigma_1(\Sigma) = 1$.

- In higher dimensions, the Clifford hypersurfaces in $\mathbb{S}^n$ have index $n+2$ and are conjectured to be only closed minimal hypersurfaces in $\mathbb{S}^n$ of index $n+2$. In contrast Smith et al. recently proved that the higher dimensional free boundary minimal catenoids Σ^{n-1} in $\mathbb{B}^n$ have suprisingly high index.
- En general, if Σ^{n-1} is a free boundary minimal hypersurface in $\mathbb{B}^n$ of index n+1, then the first Steklov eigenvalue $\sigma_1(\Sigma) = 1$.
- If Σ^2 is a free boundary minimal annulus in $\mathbb{B}^3$ of index 4, then Σ is a critical catenoid.
  In fact, ind$(\Sigma) = 4 \Rightarrow \sigma_1(\Sigma) = 1$. ($\Sigma$ is a FBMS annulus ; the coordiante functions are first Steklov fucntions.) By Theorem of Fraser-Schoen Σ is the critical catenoid.

Conjecture

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It suffices to prove that any free boundary minimal surface in $\mathbb{B}^3$ of index 4 is homeormorphic to an annulus.

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Suppose $\Sigma^2 \subset \mathbb{B}^3$ be a properly immersed free boundary minimal surface with Morse index 4. Then $u = \langle X, N \rangle$ is positive every inside.

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Theorem (Tran - 2020)

The Morse index of a free boundary minimal annulus is equal to 4 if and only if it is the critical catenoid.

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We remember that Sargent and Ambrozio et al has proved index estimates

Theorem (Sargent - 2016, Ambrozio, Carlotto, Sharp - 2017)

If Σ is a free boundary minimal surface in $\mathbb{B}^3$ of genus γ with k boundary components, then

$$ind(\Sigma) \geq \frac{1}{3}(2\gamma + k - 1)$$

Analogo of Savo's index estimates for closed minimal hypersurfaces.

Theorem (Chu - 2023)

There exists in the Euclidean unit $3-$*ball an embedded free boundary minimal surface with genus* 0 *or* 1*, Morse index 4 or 5, and area in the range* $(\pi, 2\pi)$ *that is not be equatorial disk or critical catenoid.*

Conjecture (Chu - 2023)

This surface has index 5.

- Fraser, Ailana, et al. Geometric Analysis. Springer International Publishing, 2020.
- Lima, V., Menezes, A. (2021). A two-piece property for free boundary minimal surfaces in the ball. Transactions of the American Mathematical Society, 374(3), 1661-1686.
- Li, M. (2019). Free boundary minimal surfaces in the unit ball: recent advances and open questions. arXiv preprint arXiv:1907.05053.
- Chu, A. C. P. (2023). A free boundary minimal surface via a 6-sweepout. The Journal of Geometric Analysis, 33(7), 230.
- Tran, H. (2020). Index characterization for free boundary minimal Surfaces. Communications in Analysis and Geometry, 24.
- Many papers by Fraser-Schoen about free boundary problems conditions.
- Many others ...

1. Lecture 1
   - (Brief) Motivation to study Differential Geometry. (done)
2. Lecture 2
   - Free boundary minimal or CMC surfaces. Gap results. (done)
3. Lecture 3
   - Free boundary CMC (hyper)surface in the ball. Stability. (done)
4. Lecture 4
   - Some characterization of the critical catenoid. (done)

- This is a mini-course at master/beginning of PhD level.
  1. Some classic results. ✓
  2. Free boundary CMC (minimal) (hyper)surfaces in the ball. ✓
  3. Gaps results. ✓
  4. Stability. ✓
  5. Index. ✓
  6. The Steklov Eigenvalue Problem. ✓
  7. Some characterization of the critical catenoid. ✓
  8. Some open problems. ✓

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- All the people that help me to do this possible.
- My country, Brazil, that invested in my academic training with grants (CAPES and CNPq). Without this support, certainly I would not be here.

Thank you for your attention!

Grazie mille!, ¡Muchas gracias!, Merci beaucoup!, Muito obrigada!