CIRM – A crash course in Knot Theory

Lecture I : March 4, 2024

History

1870 – 1890 : "Vortex Atom Theory" by Lord Kelvin (William Thomson)

Hypothesis : An atom is a vortex in the aether.

Tait : A systematic classification of knots up to 10 crossings.

20th century :

Poincaré : Fundamental group,

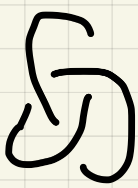
Tietze : Knot group

Reidemeister, Alexander ...

defn: A knot is a differentiable embedding of S^1 into $\mathbb{R}^3$ (or S^3).

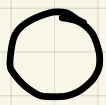
We work with "tame knots"

def: A knot is unknot if it bounds a disk.



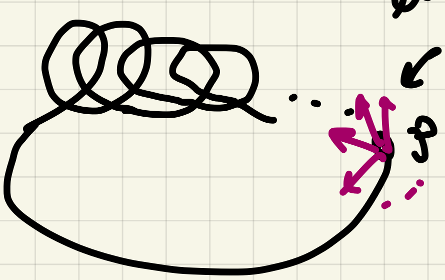
TREFOIL

FIGURE 8

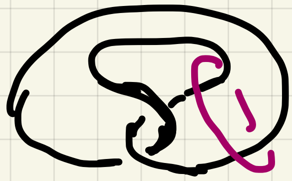


UNKNOT

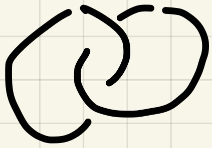
not with "wild knots" :



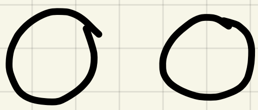
defn: A link $\mathcal{L} \subseteq \mathbb{R}^3$ is a disjoint union of knots.



WHITEHEAD LINK



HOPF LINK



UNLINK with 2 components

We consider smooth deformations on the set of all knots in $\mathbb{R}^3$.

defn: Let K_1, K_2 be two knots in $\mathbb{R}^3$.
An ambient isotopy of $\mathbb{R}^3$ taking K_1 to K_2 is a diff'ble map

$$F: \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3 \quad \text{s.t}$$

i-) $F(x,t) = f_t(x) : \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeo. $\forall\, t \in [0,1]$.

ii-) $F(x,0) = id : \mathbb{R}^3 \to \mathbb{R}^3$.

iii-) $f_1(K_1) = K_2$.

NOTE: Two smooth knots are equivalent if $\exists$ an orientation pres. homeo. $h: \mathbb{R}^3 \to \mathbb{R}^3$ taking one to another.

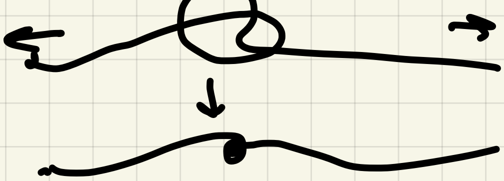
ex:

Q. Why ambient isotopy? Assume isotopy

$$F: S^1 \times [0,1] \to \mathbb{R}^3$$

$F(x,0) = K_1$, $F(x,1) = K_2$

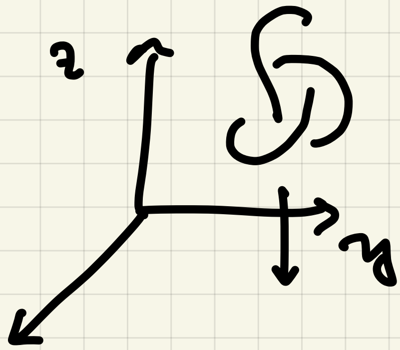
$F(x,t)$, an embedding $\forall\, t$.



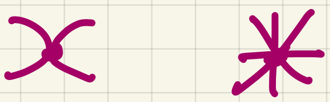
NOTE: Any embedding of S^1 into $\mathbb{R}^4$ is trivial upto ambient isotopy of $\mathbb{R}^4$.

[Proved by using general positioning, which also allows to show any two embeddings of an n-manifold into $\mathbb{R}^{2n+2}$ are isotopic.]

KNOT DIAGRAMS



{ not allowed:


}

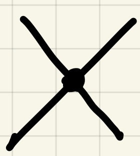
"the universe of the trefoil knot"

an immersed closed curve in $\mathbb{R}^2$

or

a 4-regular graph with n vertices with $n+2$ "regions"

We allow only transversal intersections:



n vertices $\to$ $2n$ edges

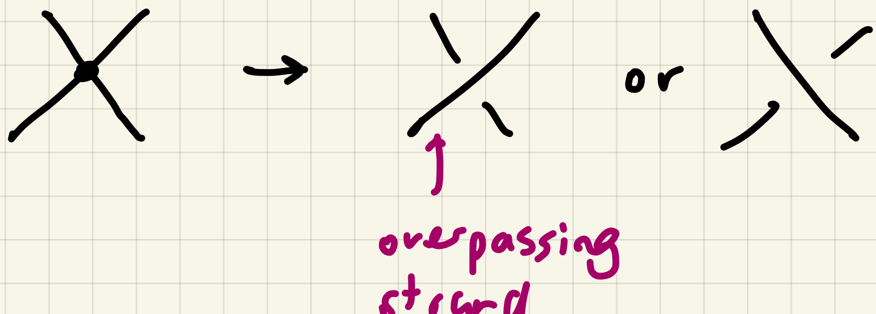
Euler's Formula:

$$n - 2n + f = 2$$

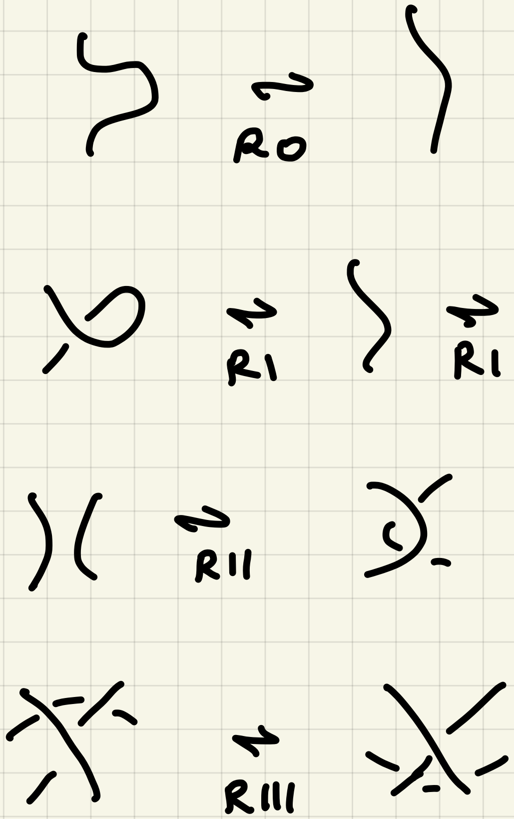
$$\Rightarrow f = n + 2$$

defn:

A knot diagram is a generic immersion of S^1 with:



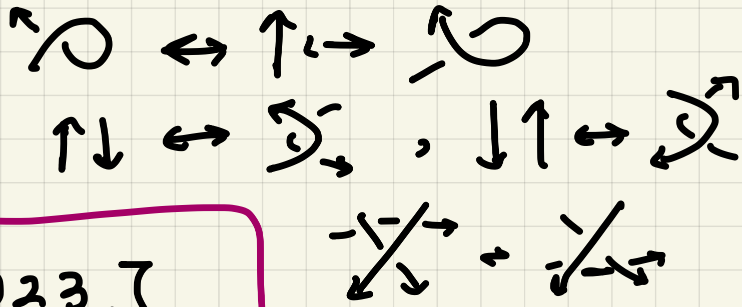
Reidemeister Moves



"planar isotopy" move

NOTE: We can consider an orientation on a knot diagram.

Thm [Polyak]: Oriented R-moves are generated by



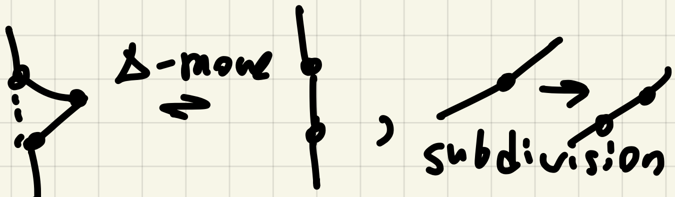
Thm [Reidemeister, 1923]

Two knot diagrams K_1, K_2 represent the "same" knot in $\mathbb{R}^3$ if and only if K_1 is related to K_2 by a sequence of Reidemeister moves.

Pf: If: easy.

Only if: i.) Consider a knot as polygonal, so that it consists of a finite number of line segments.

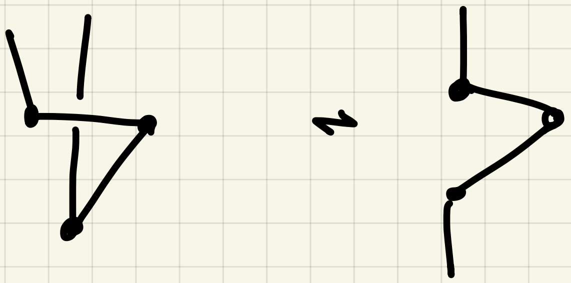
ii.) Define Δ-moves:



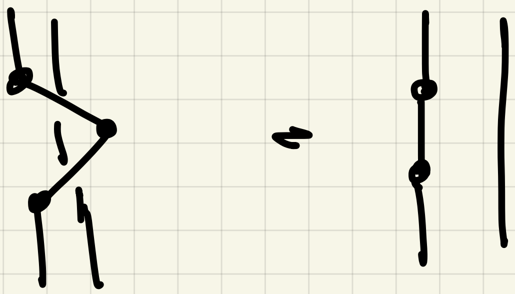
iii-) Consider projections of Δ-moves.

Apply an induction argument on the number of strands intersecting the Δ-move region:

$n = 1$:

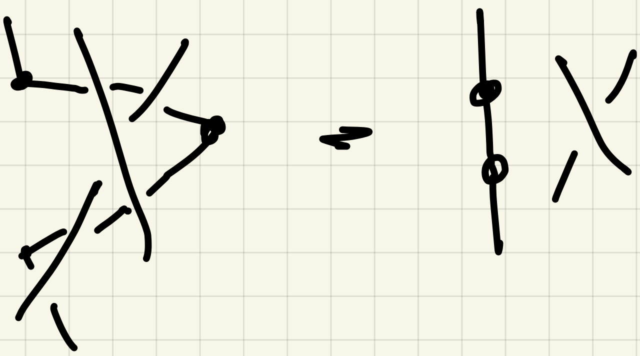


$\Rightarrow$ RI



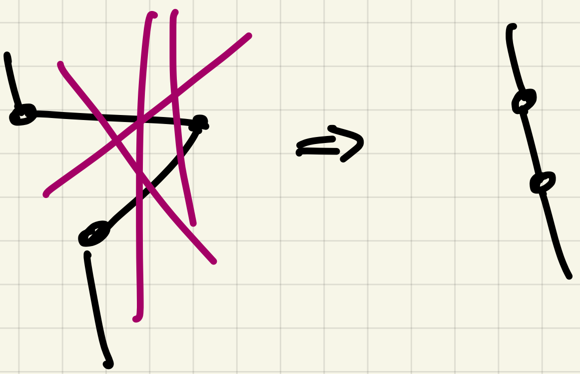
$\Rightarrow$ RII

$n = 2$



$\Rightarrow$ RIII

$n > 2$:



$\Rightarrow$ A sequence of subdivisions and RI, II, III moves.

Lecture 2 : March 4, 2024

Knot Invariants :

MAIN PROBLEM : DISTINGUISH KNOTS.

↳ DETECT UNKNOT.

1-) Crossing number

def : K, a knot (or link).

cr (K) := min. # of crossings that diagrams of K admit.

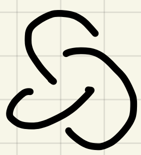
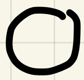
A diagram of K is minimal if it attains the crossing number of K.

minimal diagrams

cr () = 3

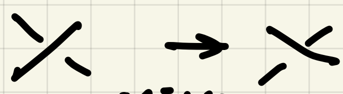
Claim : The trefoil knot is the smallest non-trivial knot.

Pf. This needs to be proven.

Q.  $\overset{\sim}{?}$ 

NOTE : Trefoil knot is chiral.

Figure - 8 knot is amphi-chiral.

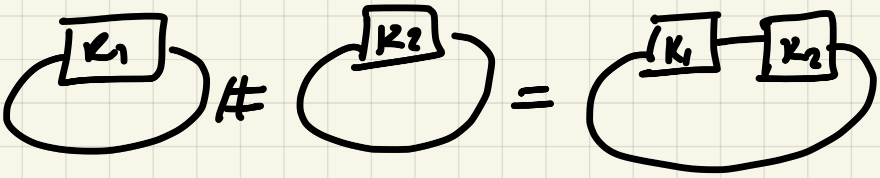


Mirror image

A Solved Conjecture : [Tait 's Conj.]

"A reduced alternating diagram of a knot is minimal. Proven by Kauffman & Thistlethwaite in 1987 by the bracket.

Open Problem :



$(\mathcal{K}, \#)$ is a comm- monoid.

$$cr(K_1 \# K_2) = cr(K_1) + cr(K_2).$$

Clearly $cr(K_1 \# K_2) \leq cr(K_1) + cr(K_2)$

$\geq$?

Marc Lackenby :

$\exists$ a constant $N > 1$ s.t

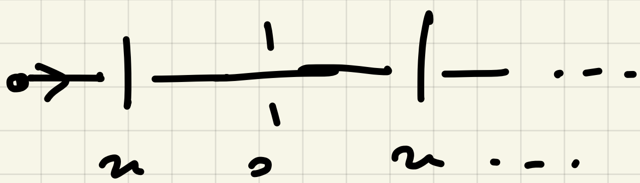
$$\frac{1}{N}\left(cr(K_1) + cr(K_2)\right) \leq cr(K_1 \# K_2).$$

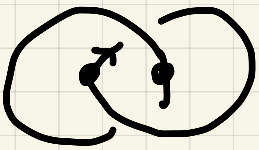
Thm : [Kauffman, Murasugi, Thistlethwaite]

Let K_1, K_2 be two alternating knots.

$$cr(K_1 \# K_2) = cr(K_1) + cr(K_2).$$

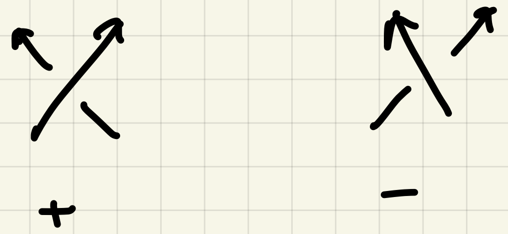
alternating link





Linking Number

Sign:

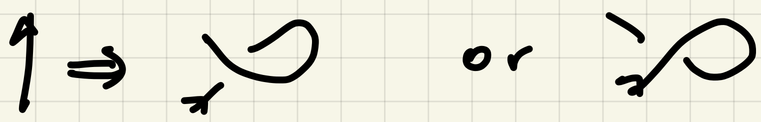


def: Let $\vec{L}$ be a link with 2 components, α and β.

$$lk(\vec{L}) = \frac{1}{2} \sum_{c \in \alpha \cap \beta} \text{sign}(c).$$

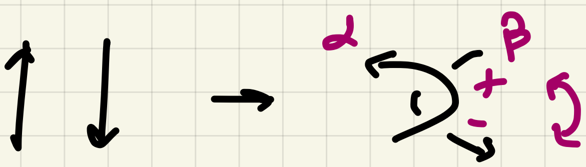
Thm: The linking number is an oriented link invariant.

Pf: Check the invariance under oriented Reidemeister moves.



a kink is added to a local part of a single component

$\Rightarrow$ lk doesn't change.



"opposite signs cancel each other"

RIII - check exercise.

NOTE:

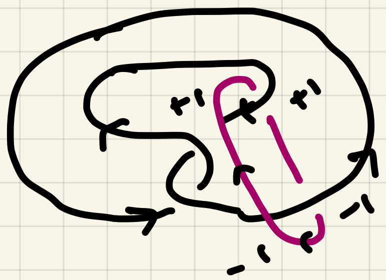
Interpretation of the linking number in $\mathbb{R}^3$ was given by Gauss ~ 19th century as a double integral in the search for the terrestrial magnetic potential.

Check Renzo Ricca & Bernardo Nipoti

"Gauss linking number revisited" - 2011

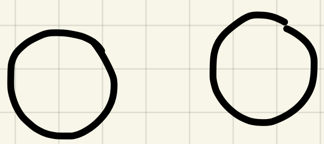
Q.

RK



$= 0$

S ?

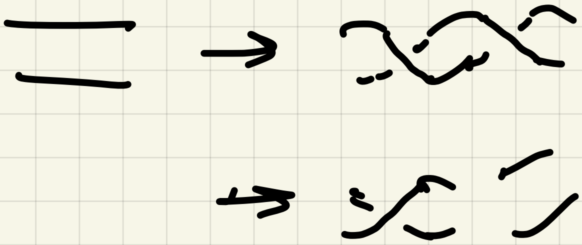


- Writhe $\vec{K}$ oriented link diagram.

$$wr(\vec{K}) = \sum_{c \in \mathcal{C}(K)} \text{sign}(c).$$

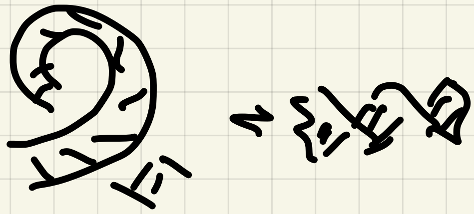
Writhe is invariant under RII and RIII

But not under RI. [Regular isotopy invariant of framed knots.]

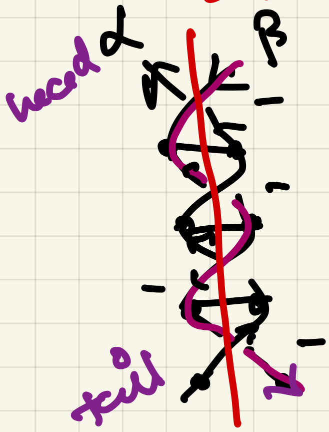


left handed

right handed

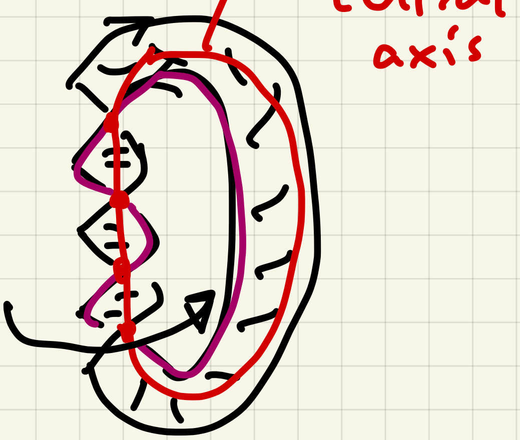


an application



DNA helix

close each strand to itself



a link!

White & Calugareanu:

\# int. in central axis

winding of α, β around each other

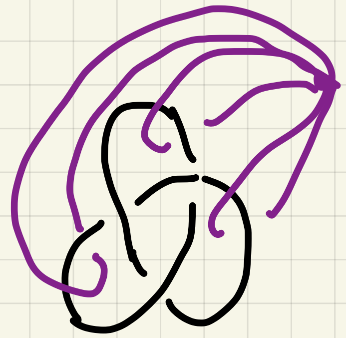
$$\boxed{\mathcal{L}K(\alpha, \beta) = \text{Writhe} + \text{Twist}}$$

$$-2 = \emptyset - 2$$

Knot Group

Tietze $\to$ Gordon & Luecke

$\underline{\underline{K}}$



Each arc generates a loop in $\mathbb{R}^3 - K$.

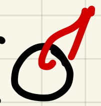
$S^3 - K$ is a compact 3-manifold with $\partial = K$.

Immediate Observation:

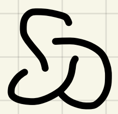
If $K_1 \sim K_2$ then $\mathbb{R}^3 - K_1 \cong \mathbb{R}^3 - K_2$.

Therefore $\pi_1(\mathbb{R}^3 - K_1) \cong \pi_1(\mathbb{R}^3 - K_2)$

defn: The knot group of a link, $\pi(K)$ is the fundamental group of $\mathbb{R}^3 - K$.

$\pi($  $) = \mathbb{Z}$. [∞ cyclic group]

$\pi($  $) = ?$ hard to compute.

Can it be  $\sim$  ?

Thm: (Gordon & Luecke)

upto mir. sym.

If $\mathbb{R}^3 - K_1 \cong \mathbb{R}^3 - K_2$ then $K_1 \sim K_2$.

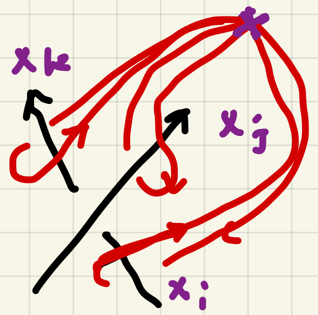
(Whitten)

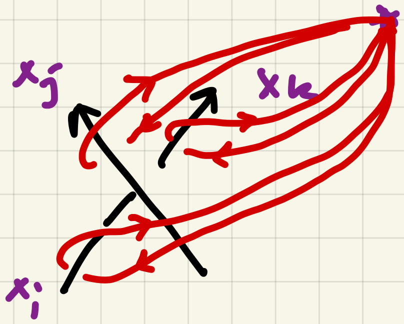
Thm: K_1, K_2 prime knots. If $\pi(K_1) \sim \pi(K_2)$ (isomorphic) then $\mathbb{R}^3 - K_1 \cong \mathbb{R}^3 - K_2$.

Corollary: The knot group is a complete invariant of prime knots up to mirror symmetry.

A planar presentation of the knot group

K, oriented:





$\boxed{x_j^{-1} x_i x_j = x_k}$ $\boxed{x_j x_i x_j^{-1} = x_k}$

Arcs $\Rightarrow$ generators

Crossings $\Rightarrow$ relations

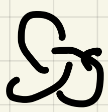
gen. by relations

$$\pi(K) = \langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle$$

Quotient of Free group on $x_1, \dots, x_m$ by the normal subgroup

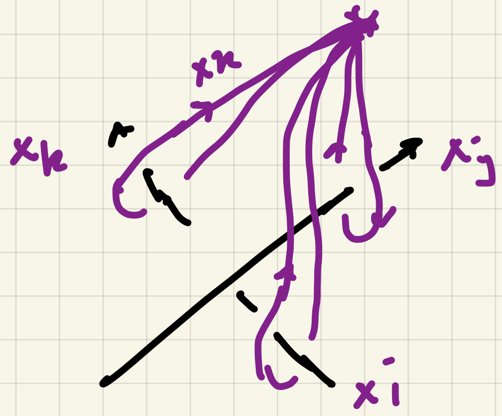
exercise: Let K be a knot with n crossings. In its knot group presentation, at most $n-1$ relations are needed.

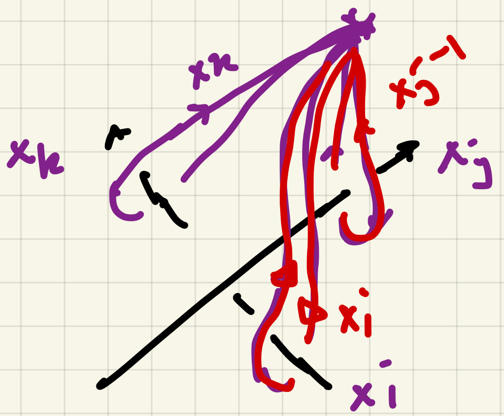
exercise: Find $\pi($  $)$.

$\hookrightarrow$ Prove that  is non-trivial.

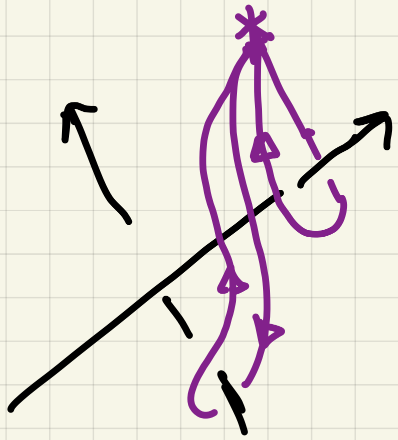
Exercise Revealed [Crossing relations:]

$x_j^{-1} x_i x_j x_k^{-1} = 1$

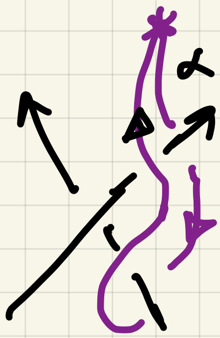




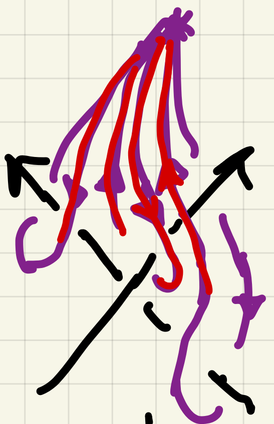
$\xrightarrow{x_j^{-1} x_i}$



homotopy

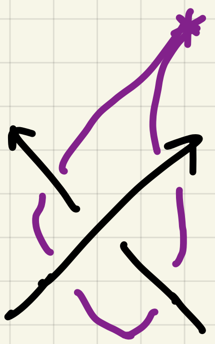


$\xleftarrow{\alpha x_j x_k^{-1}}$



homotopy

(pull down the red parts)



trivial loop!

$\therefore x_j^{-1} x_i x_j x_k^{-1} = id$!

NOTE: $\exists$ nonequivalent composite knots with isomorpic knot groups.

ex: Granny knot:

K

K_1, K_2 both right-handed trefoil [or both left-handed!]

Square knot

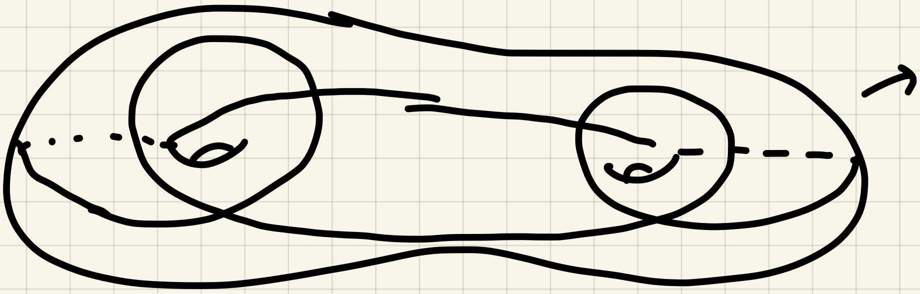
$\tilde{K}$

K_1: right-handed trefoil

K_2: left-handed trefoil

$$\pi(K) = \pi(\tilde{K}).$$

But $V_K \neq V_{\tilde{K}}$! (exercise.)



Kishino knot

— END —

Lecture 3: March 5, 2024

The Jones polynomial

1) The first knot polynomial : Alexander Poly.
~1920s

2) The Alexander-Conway poly. John Conway
"skein relation" ~1960s

3) The Jones polynomial Vaughan Jones
~1980s.

"von Neumann algebras
$\updownarrow$
braid group" $\Longrightarrow$ TQFT

$$\begin{array}{ccc} B_n & \xrightarrow{\rho} & TL_n \\ & \searrow^{\langle\ \rangle} & \downarrow tr \\ & & \mathbb{Z}[t, t^{-1}] \end{array}$$

got the Fields Medal in 1991
with Edward Witten "path integral reformulation" Feynman

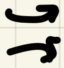
$\Downarrow$

The Bracket polynomial : Louis Kauffman

Axiomatic defn: Let K be an oriented knot/link. The Jones polynomial is the unique Laurent polynomial in $\sqrt{t}$ that satisfies

i. If $K \sim K'$ then $V_K(t) = V_{K'}(t)$.

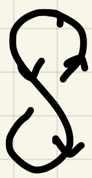
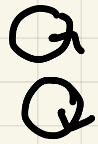
ii. $V_{\bigcirc}(t) = 1$

iii.) $t^{-1}\, V$  $-\, t\, V$  $= \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V$  .

skein relation

Possible to compute the Jones polynomial by using the axioms.

By AIII :

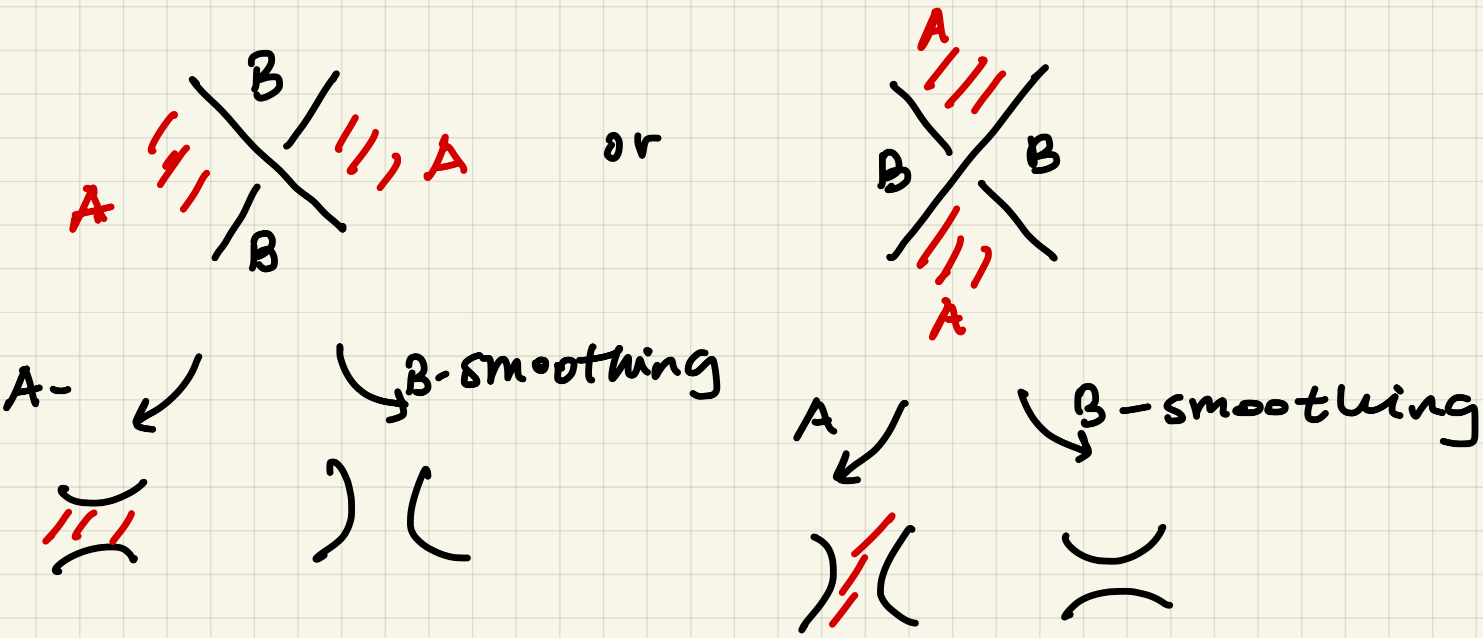
$t^{-1}\, V$  $-\, t\, V$  $= \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V$  .

$$\underset{A\,I,\,II}{\Rightarrow} \quad \frac{t^{-1} - t}{\sqrt{t} - \frac{1}{\sqrt{t}}} = V_{\bigcirc\,\bigcirc} .$$

Q. How do we know such a polynomial exist?

The Bracket Polynomial

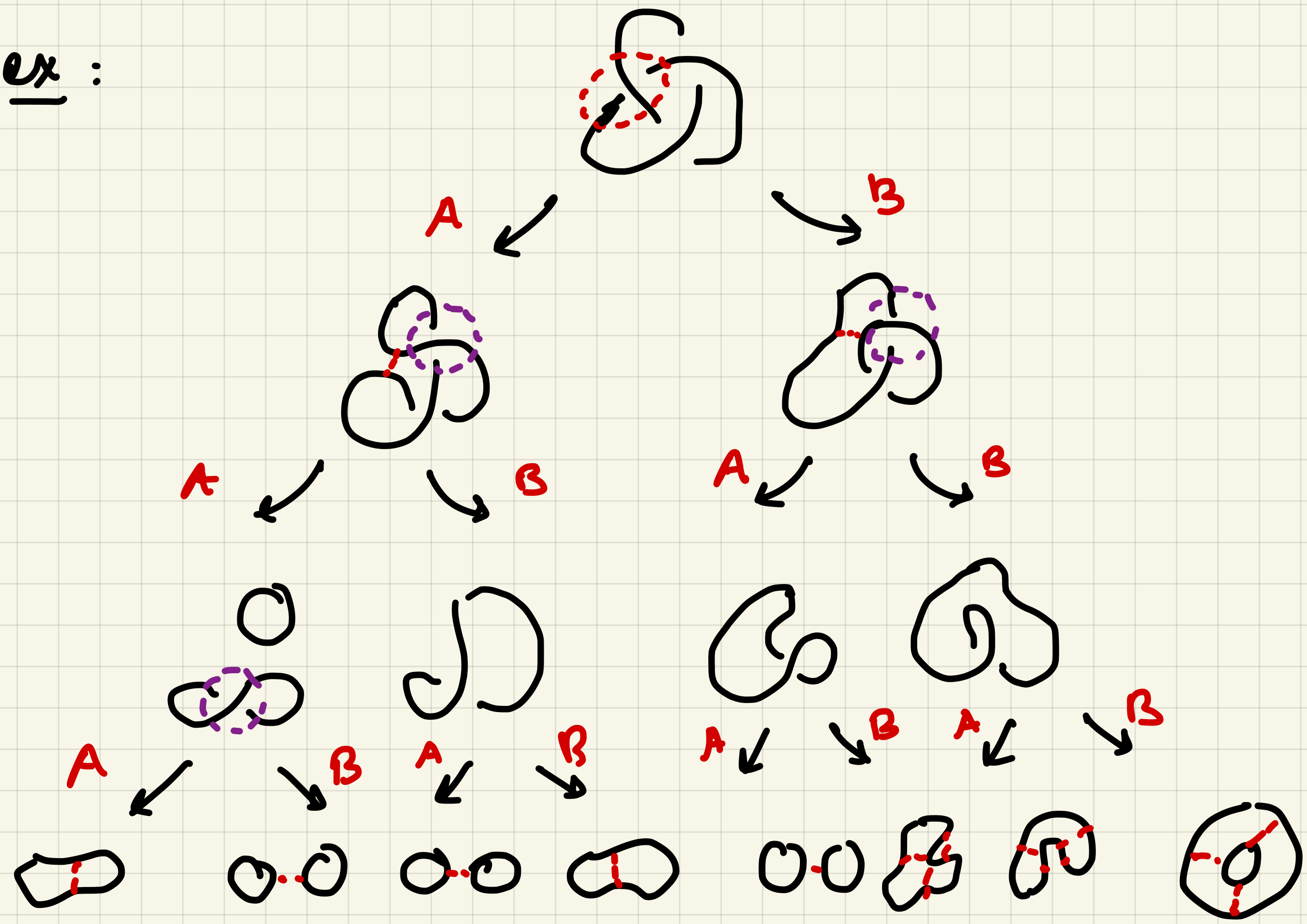
K, a non-oriented diagram of a knot



→ Smooth all crossings of K in 2 possible ways

⇒ 2^n loop configurations we have.

ex:

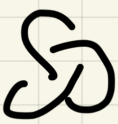
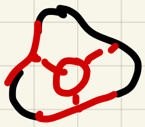


def : The final descendents of K after smoothing each crossing out are called states of K.

Aim: To construct a link invariant by taking an "average" over states of the link.

Define the weight of s:

$$\langle K \mid s \rangle := \prod \text{smoothing labels to obtain } s$$

$\langle$  $\mid$  $\rangle = B^3$

The norm of s:

$$\| s \| = \#\text{ loops} - 1 .$$

defn: The bracket polynomial of K is given as

$$\langle K \rangle (A, B, d) = \sum_{s \in S} \langle K \mid s \rangle . d^{\| s \|}$$

NOTE: $\langle O \, O \rangle = d$

$$\Rightarrow \langle O \, K \rangle = d \, \langle K \rangle .$$

Thm: The bracket polynomial of K is invariant under RII and RIII-moves if $B = A^{-1}$ and $d = -A^2 - A^{-2}$.

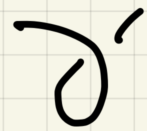
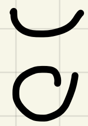
Proof:

RII-invariance requires $AB = 1$ and $d = -A^2 - A^{-2}$.

Sufficient condition for RIII-invariance.

$\Rightarrow \langle K \rangle \in \mathbb{Z}[A, A^{-1}]$.

What about RI-moves?

$\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

$= A d \langle \smile \rangle + A^{-1} \langle \smile \rangle$

$= \left(A(-A^2 - A^{-2}) + A^{-1}\right) \langle \smile \rangle$

$= (-A^3 - \cancel{A^{-1}} + \cancel{A^{-1}}) \langle \smile \rangle$

$= -A^3 \langle \smile \rangle$

it changes by A^3

$\Rightarrow$ Normalize $\langle K \rangle$ by the writhe factor:

$$(-A^3)^{-w(K)},$$

where K is given an orientation now.

$$w(\quad) = w(\quad) - 1$$

$$w(\quad) = w(\quad) + 1$$

$$(-A^3)^{-(w+1)} \langle \quad \rangle = (-A^3)^{-(w+1)} \cdot (-A^3) \langle \quad \rangle$$

$$\Rightarrow (-A^3)^{-(w+1)} \langle \quad \rangle = (-A^3)^{-w} \cdot \langle \quad \rangle$$

Define $f_K(A) = (-A^3)^{-w} \langle K \rangle$.

Corollary: $f_K(A)$ is an oriented link invariant.

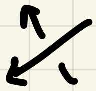
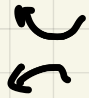
Proposition:

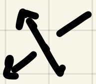
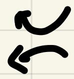
$$\langle \quad \rangle = A \langle\,)\,(\,\rangle + A^{-1} \langle \quad \rangle.$$

Pf: $\text{States}\{\quad\} = \underbrace{\text{States}\{\,)\,(\,\}}_{S_1} \sqcup \underbrace{\text{States}\{\quad\}}_{S_2}$

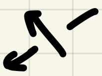
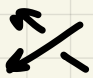
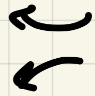
$$\Rightarrow \langle \quad \rangle = \sum_{s \in S(\quad)} \langle K | s \rangle d^{\|s\|} = A \sum_{s} \langle K|s \rangle d^{\|s\|} + A^{-1} \sum' \langle K|s \rangle d^{\|s\|}.$$

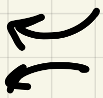
Thm: $f_K(t^{-1/4}) = V_K(t)$.

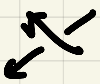
Pf: $A^{-1}\,/\,\langle$  $\rangle = A\,\langle\,)(\,\rangle + A^{-1}\langle$  $\rangle$

$A\,/\,\langle$  $\rangle = A\,\langle$  $\rangle + A^{-1}\langle\,)(\,\rangle$

$\Rightarrow$

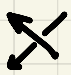
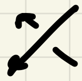
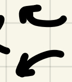
$A\,\langle$  $\rangle - A^{-1}\langle$  $\rangle = (A^2 - A^{-2})\,\langle$  $\rangle$.

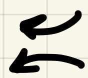
let $w($  $) = w$

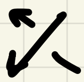
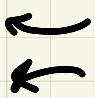
then $w($  $) = w+1$ let $-A^3 = \alpha$

$w($  $) = w-1$

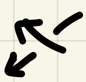
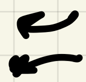
Then, multiply each term by α^{-w}:

$A\cdot\alpha\,\alpha^{-(w+1)}\langle$  $\rangle - A^{-1}\alpha^{-1}\alpha^{-(w-1)}\langle$  $\rangle = (A^2 - A^{-2})\,\alpha^{-w}\langle$  $\rangle$

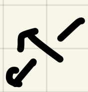
f  f  f 

$\Rightarrow\; -A^4 f$  $+ A^{-4} f$  $= (A^2 - A^{-2})\, f$ 

let $A = t^{-1/4}$.

$\Rightarrow\; -t^{-1} f$  $+ t\, f$  $= (t^{-1/2} - t^{1/2})$ 

M. by -1

$\Rightarrow\; t^{-1} f$  $- t\, f$  $= (t^{1/2} - t^{-1/2})$  .

$\square$

NOTE: This equality also gives a proof for the existence of the Jones polynomial.

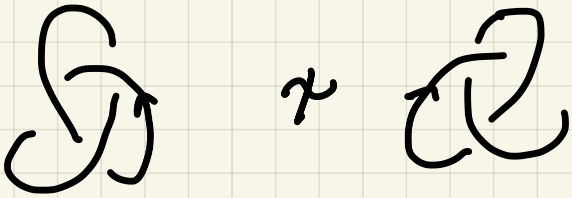
Prop: Let K be an oriented knot and K^* be its mirror image.

Then

$$\langle K^* \rangle (A) = \langle K \rangle (A^{-1}).$$

But reversing the orientation doesn't change $\langle . \rangle$!

Corollary:



Proposition: Let K and K' be oriented links s.t. K' is obtained by reversing the orientation of a component K_1 of K.

Let $\lambda = lk(K_1, K - K_1)$ denote the total linking number of K_1 with the other components.

Then

$$V_{K'}(t) = t^{-3\lambda} V_K(t).$$

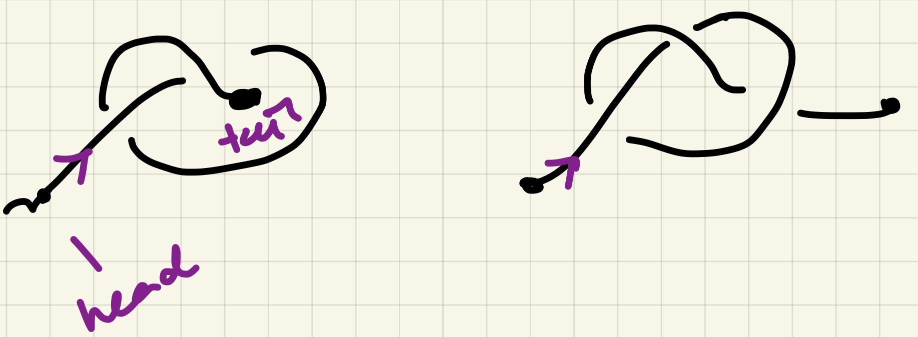
Pf: exercise.

Lecture 4: March 5, 2024

An oriented generalization for knotoids

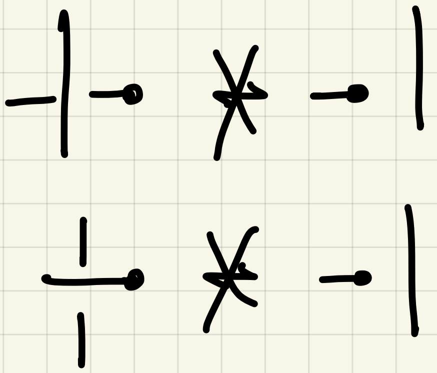
[Turaev, 2012]

def: A knotoid diagram is an [generic] immersion of $[0,1]$ into a surface.



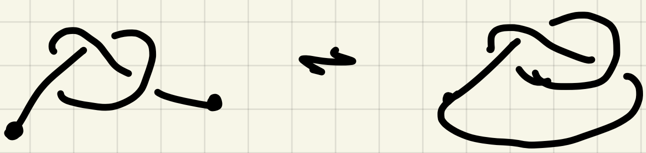
Endpoints may appear in different regions unlike the case of 1-1 tangles.

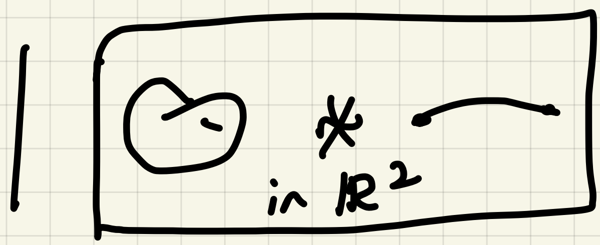
Forbidden Moves:



$\{\text{Knotoid diagrams}\}/_\sim$

$K_1 \sim K_2$ if $\exists$ a finite seq. of R-moves taking K_1 to K_2

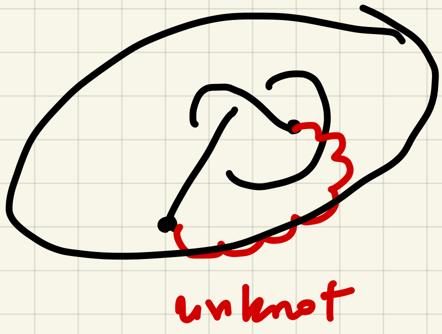


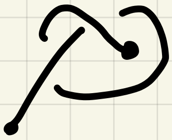


$\{\text{Knots in } S^3\} \hookrightarrow \{\text{Knotoids in } S^2\}$.

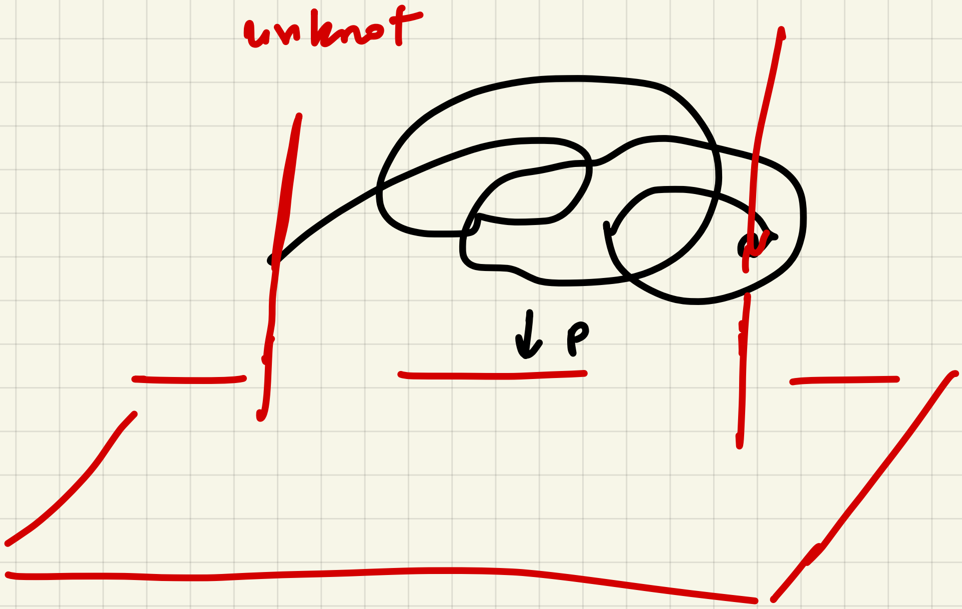
Application:

→ Utilized in protein modelling to understand the geometrical structure.



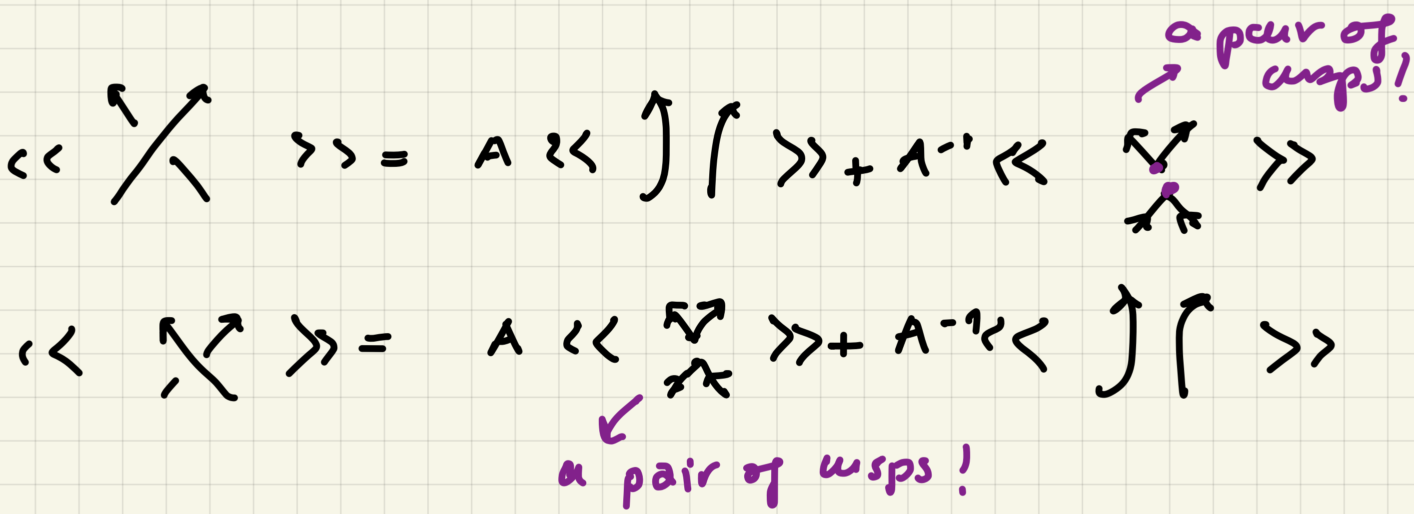


is nontrivial!



The Arrow Polynomial:

It's a generalization of the bracket polynomial, based on oriented smoothing:



⇒ Smooth all crossings of an oriented knot / knotoid diagram.

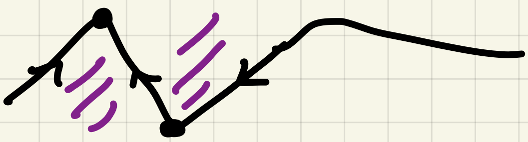
⇒ Each state of a knotoid diagram contains a number of loops and exactly one "long" component with endpoints.

⇒ Consider each state up to reduction rules:

If

" inside of cusps are on the same side"

If



⇒ keep this pair ! of cusps

" inside of cusps lie on opposite sides"

Prop: All cusps on a loop component in a state are reduced.

(outline:)

Pf: Observation I: All cusps in a state are paired.

Observation II: ∃ an even number of cusps on a state component.

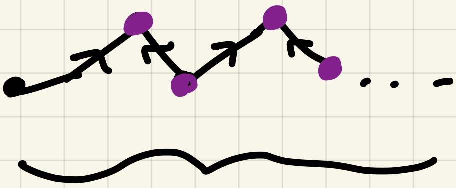
Then by the Jordan curve thm, all cusps are necessarily reduced on a loop component. □

Thus,

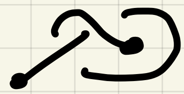
$$\langle\langle \bigcirc \rangle\rangle = 1$$

$$\langle\langle \bigcirc \; K \rangle\rangle = (-A^2 - A^{-2}) \langle K \rangle$$

a new variable !

$\langle\langle$  $\rangle\rangle = \lambda^n$

long component with $\underline{\underline{n}}$ "zigzags"

exercise : Calculate $\langle\langle$  $\rangle\rangle$.

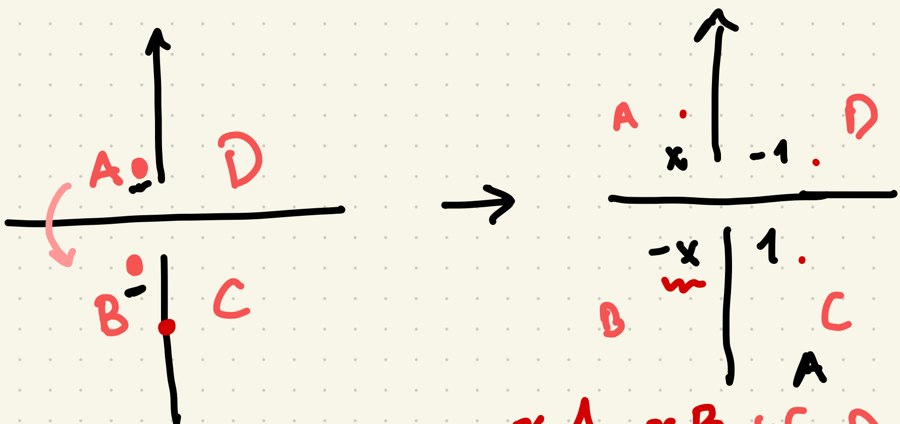
Alexander Polynomial

· 1928. J.W. Alexander

"Topological Invariants of knots and links
Trans. Amer. Math. Soc. 30 (1928) 275-306."

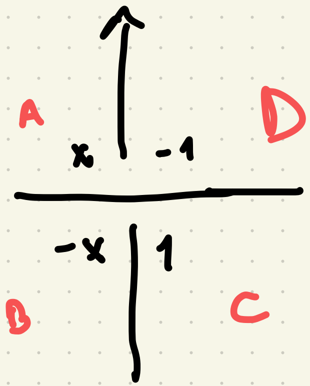
Given an oriented knot/link diagram.

4 local regions around a crossing:



$$xA - xB + C - D = 0$$

1. At each crossing two dots are placed just to the left of the underpassing arc, one before, one after the overcrossing.
2. Associate labels to each local region.

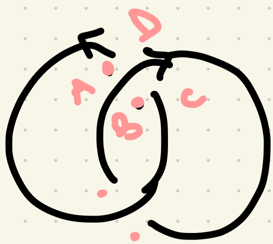


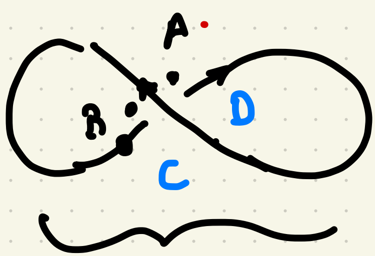
$$\longleftrightarrow \quad xA - xB + C - D = 0$$

Note: If some of the regions are the same at the crossing, then the equation is simplified.

For instance if $A = D$ we have

$$xA - xB + C - A = 0$$





$cr_1 : xA - xB + C - D = 0$

$cr_2 : xA - xD - B + C = 0$

$cr : xA - xB - D - A = 0$

Each crossing gives an equation involving the regions of the diagram $\vec{K}$.

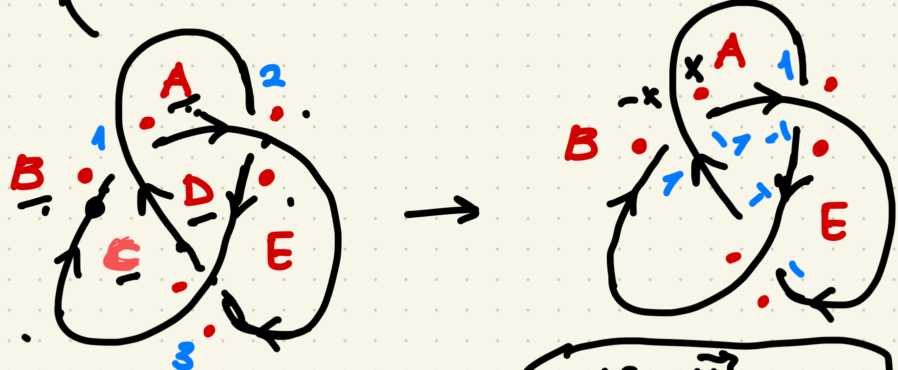
$\Rightarrow$ Construct a matrix for $\vec{K}$,

M_K s.t rows $\leftrightarrow$ equations

columns $\leftrightarrow$ regions of $\vec{K}$

left-handed

Ex:



1: $xA - xB + C - D = 0$

2: $xE - xB + A - D = 0$

3: $xC - xB + E - D = 0$

* If $\vec{K}$ has n crossings, then K admits $n+2$ regions.

↳ Ex.

$$\Rightarrow M_K = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} x & -x & 1 & -1 & 0 \\ 1 & -x & 0 & -1 & x \\ 0 & -x & x & -1 & 1 \end{pmatrix} \end{array}$$

Delete two columns that correspond to adjacent regions to obtain a square matrix; $M_K(\dot{R}_1, \dot{R}_2)$.

definition: The Alexander polynomial of K, $\Delta_K(x)$ is given by

$$\underbrace{\Delta_K(x)}\ \doteq\ \mathrm{Det}\left(\underline{M_K(\dot{R}_1,\dot{R}_2)}\right).$$

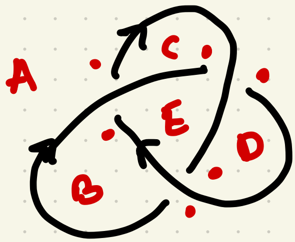
$$A \doteq B \longleftrightarrow A = \pm x^n B,\ n \in \mathbb{Z}.$$

ex:

$$M = \begin{array}{c} \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{pmatrix} x & -x & 1 & -1 & 0 \\ 1 & -x & 0 & -1 & x \\ 0 & -x & x & -1 & 1 \end{pmatrix} \end{array}$$

$$\Rightarrow M(A,B) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & x \\ x & -1 & 1 \end{pmatrix}$$

$$\Rightarrow \Delta(x) \doteq (-1+x) - x^2 = (-1 + x - x^2)\cdot x$$



1: $xA - xB - C + E = 0$

2: $xA - xC + E - D = 0$

3: $xA - xD - B + E = 0$

Right-handed trefoil

$$
M_{K^*} = \begin{array}{c} \\ \\ \end{array}
\begin{array}{ccccc} A & B & C & D & E \end{array}
$$

$$
M_{K^*} = \begin{pmatrix} x & -x & -1 & 0 & +1 \\ x & 0 & -x & -1 & +1 \\ x & -1 & 0 & -x & +1 \end{pmatrix}
$$

: Alexander matrix

$$
\det \begin{pmatrix} -1 & 0 & +1 \\ -x & -1 & +1 \\ 0 & -x & +1 \end{pmatrix} = -(-1+x) + x^2 = \underline{1 - x + x^2}
$$

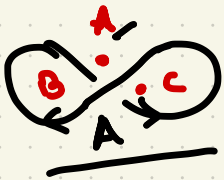
Notice that Δ  $(x) \doteq \Delta$  (x)

K, K^* → mirror image of K

$$
\boxed{\Delta_K(x) \doteq \Delta_{K^*}(x)} \quad \forall K.
$$

we have to prove this.

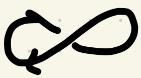
Ex:

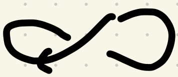


$$m_K = \begin{array}{ccc} A & B & C \\ \left(\cancel{-x}\right. & \cancel{-1} & \left.x\right) \end{array}$$

$$xC - xA + B - A = 0$$

$$\begin{array}{ccc} A & B & C \\ \left(-x-1\right. & 1 & \left.x\right) \end{array}.$$

$\Rightarrow \Delta$  $(x) \doteq 1.$

But  $\sim$ 

Then Δ  $(x) \doteq 1$

note: we have not proven yet Δ is a knot invariant.

Theorem: $\Delta_K(x)$ is a knot invariant up to $\doteq$

Proof: Check

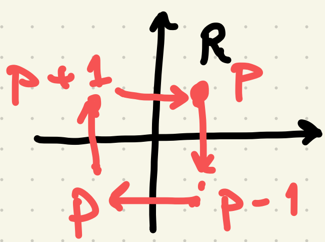
1. the poly. is well-defined, that is it is independent of the choice of adjacent regions,

2. is invariant under the Reidemeister moves up to $\doteq$

Define the index of a region:

$R \xrightarrow[i]{} p$, $p \in \mathbb{Z}$, R any chosen region.

index all regions:

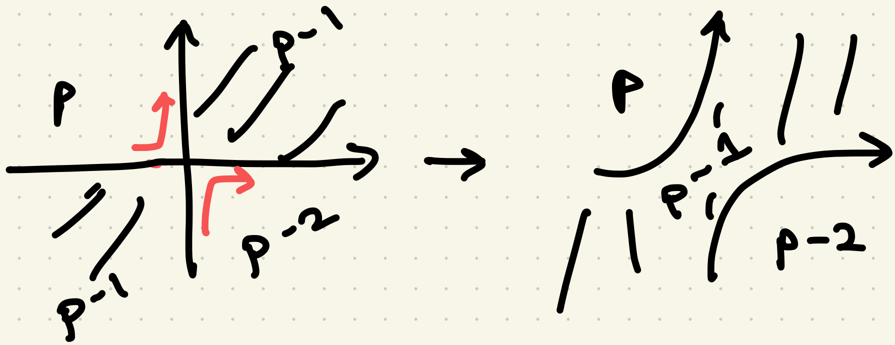


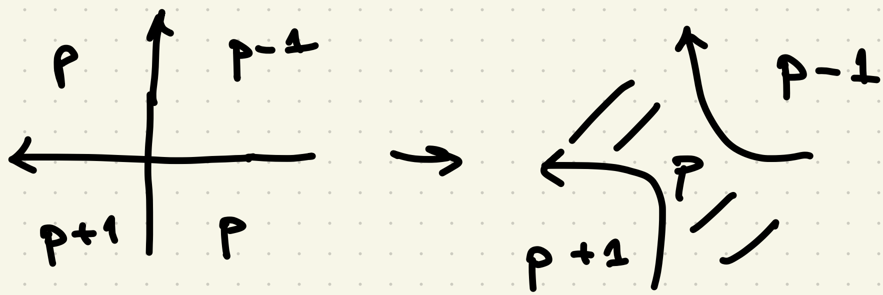
Start with any region R

Right $\to$ left $\Rightarrow p+1$

Left $\to$ right $\Rightarrow p-1$.

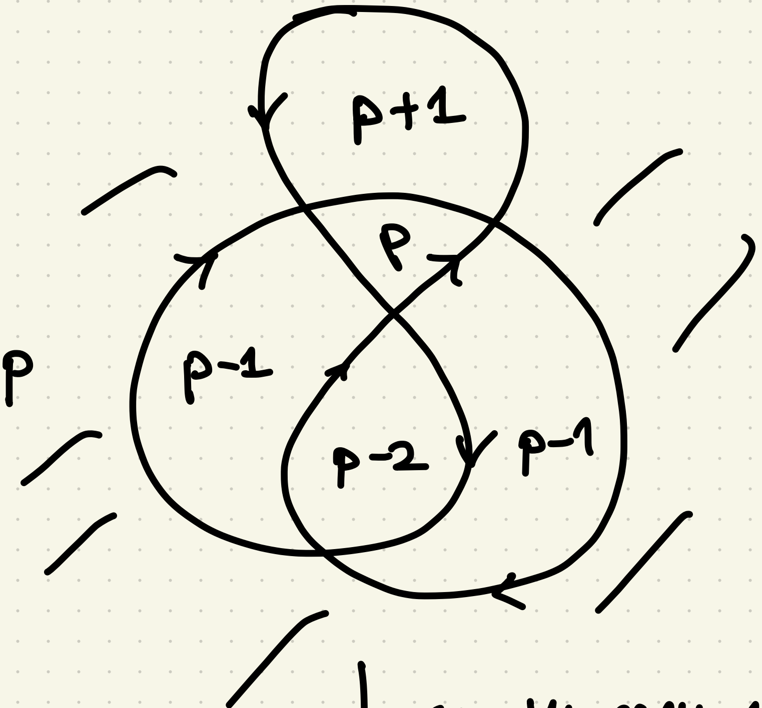
Observe that if we smooth the crossing according to the orientation on the arcs, the same indexed regions connect to be the same "region".



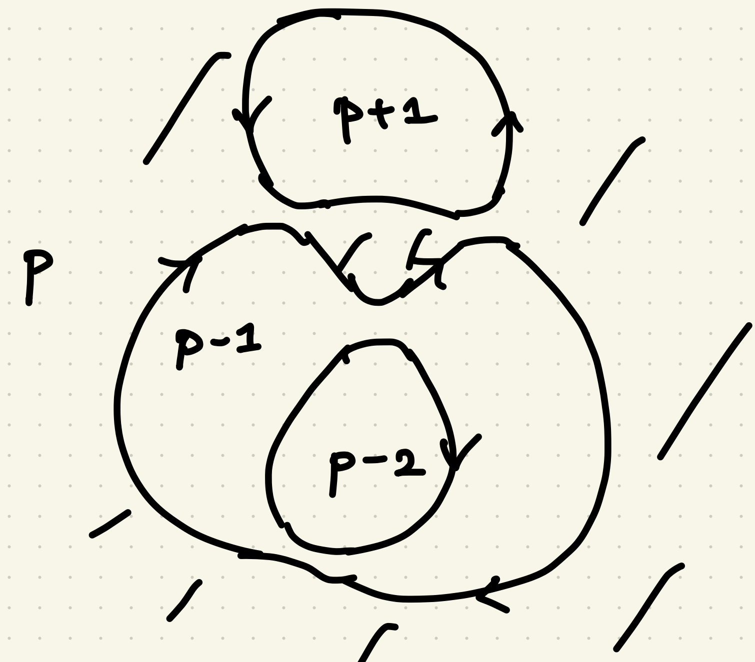


After smoothing all crossings in this way of a given oriented knot / link diagram, we obtain <u>Seifert circles</u> (simple closed curves in $\mathbb{R}^2$ with orient.) bounding the resulting regions

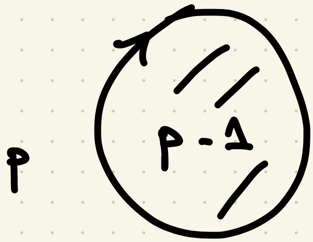
$\longrightarrow$



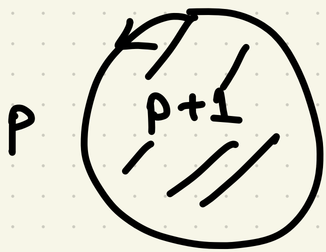
smooth each crossing
in the "oriented" way



Notice that each Seifert circle divides the plane into 2 regions by the Jordan curve theorem. Each region bdd by a Seifert circle is reached from the "outer" region either from left to right or right to left. Thus, the regions bdd by a Seifert circle are indexed uniquely by the given algorithm.



or



"Jordan curve Thm. with indices"

By the prev. observation that the same indexed regions are connected after the smoothing, and the observation above, we have a well-defined labeling for each region of K.

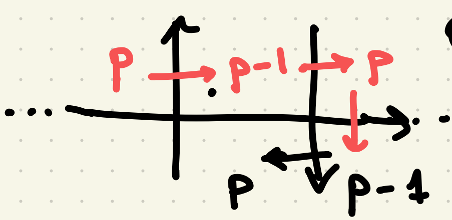
Another reasoning:

The underlying graph of a knot is a 4-valent planar graph with n vertices.

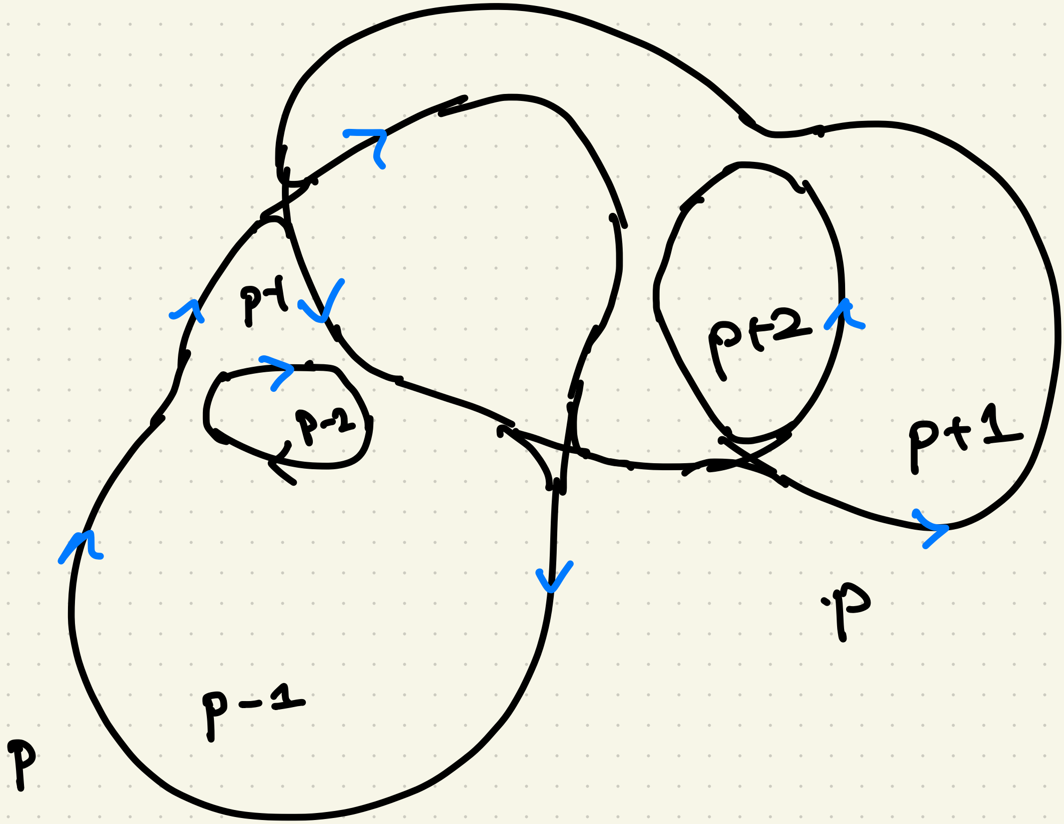
Each edge is shared by two vertices thus there are in total $2n$ edges of the graph.

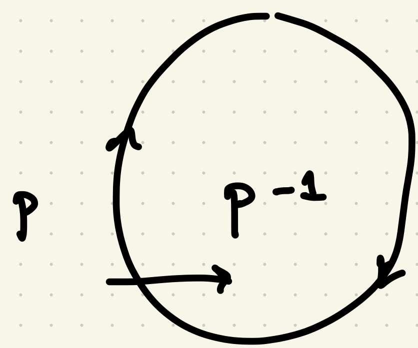
Assume the "unbounded" region of the graph is labeled by 0.
We start labelling the other regions.

Notice that from each edge we cross twice.



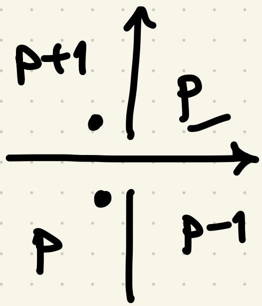
$0 \to \dots \to \underline{p+n} \to R.$


p-1
p-2
p+2
p+1
p
p-1
p

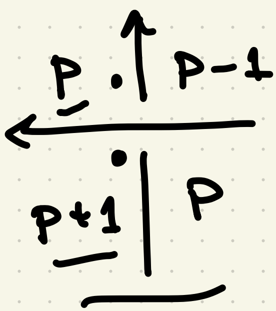

p
p-1

- All regions can be reached by crossing over the arcs of the diagram, all regions are indexed

- Two local regions are indexed the same.



→ dotted regions receive indices p and p+1.



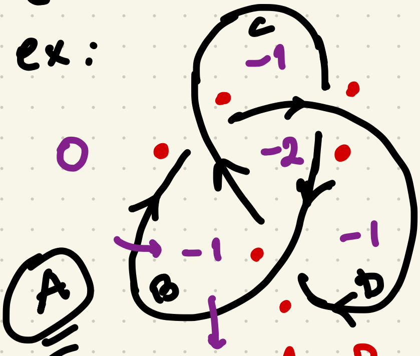
→ dotted regions recieve indices p+1 and p.

- The indices of adjacent regions differ by 1.

Proposition: Let M_K be the matrix of Alexander's equations. If we reduce M_K by deleting two columns of index p and $p+1$ then the determinants of two matrices will differ only by a factor $\pm x^k$ for any such two columns.

Proof: Let R_p denote the sum of all columns of index p. Assume the unbdd region is indexed by 0

ex:



$$R_0 = \begin{pmatrix} -x \\ -x \\ -x \end{pmatrix}, \quad R_{-1} = \begin{pmatrix} x+1 \\ x+1 \\ x+1 \end{pmatrix}$$

$$R_{-2} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$M_K = \begin{array}{c} \begin{matrix} A_0 & B_{-1} & C_{-1} & D_{-1} & E_{-2} \end{matrix} \\ \begin{pmatrix} -x & 1 & x & 0 & -1 \\ -x & 0 & 1 & x & -1 \\ -x & x & 0 & 1 & -1 \end{pmatrix} \end{array}$$

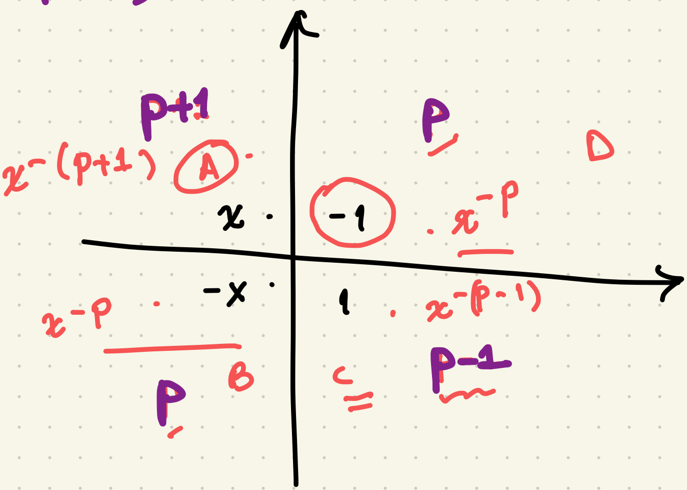
Notice:

$$\boxed{R_0 + R_{-1} + R_{-2} = \vec{0}}$$

In general $\sum_p R_p = \vec{0}$.

Claim: $\sum_p x^{-p} R_p = \vec{0}$

Pf: (outline)



$$A := x^{-(p+1)} \cdot x = \cancel{x^{-p}}$$

$$B := x^{-p} \cdot -x = \ominus\, \cancel{x^{-p+1}}$$

$$C := x^{-(p-1)} = \cancel{x^{-p+1}}$$

$$D := x^{-p} \cdot (-1) = -\cancel{x^{-p}}$$

$$+ \quad \overline{\qquad \cancel{0} \qquad}$$

trivial sum.

Notice that $\sum_p R_p$ is the sum of all columns.

Multiplying column entries in a row by x^{-p} results in

Multiply each column by x^{-p}

$$\Rightarrow \sum_p x^{-p} \cdot R_p = \vec{0},$$

$$\Rightarrow \underbrace{\sum (x^{-p} - 1)}\, R_p = \vec{0}$$

$$\left(= \sum_p x^{-p} R_p - \sum_p R_p\right)$$

In the trefoil ex., we have

$$0 = \cancel{(x^0} R_0 - \cancel{R_0}) + (x^{+1} - 1) B_{-1} + (x^{+1} - 1) D_{-1}$$

$$\rightarrow (x^{+1} - 1) C_{-1} + (x^2 - 1) R_{-2} = 0$$

That is, the terms in R_0 in the sum cancel each other out.

If r_j is a region of index p with corresponding column c_j then

$(x^{-p} - 1)\, c_j$ is expressible as a linear combination of the other columns with non zero index.

Also, the coefficients of the columns in the linear combination are of the form $-(x^{-q}-1)$ for each column of index q.

Let c_j be a column related to a p-indexed region, and
c_k be a column " " q-indexed region.

Consider the matrices $M(0, c_j)$, $M(0, c_k)$ obtained by removing the 0-indexed region (the unbdd. region) column, say c_0.

Now, notice that

$\forall\ c_j \in M_k(0, c_k)$,

$$(x^{-p}-1)\,c_j = -\sum_{c_i \in M_k(0, c_j)} (x^{-q}-1)\,c_i \, .$$

by the above observation.

Then, we find $\longrightarrow$

$(x^{-p}-1)\,\mathrm{Det}(0, c_k)$ k^{th} column deleted

$$= \mathrm{Det}\left(c_1, \dots, (x^{-p}-1)c_j, \dots, c_{k-1}, c_{k+1}, \dots, c_n\right)$$

$$= \mathrm{Det}\Big(c_1, \dots, \underbrace{-\sum_{i \in \{1,\dots,n\} - \{0,j\}} (x^{-q}-1)\, c_i}_{j^{th}\text{ column}}, \dots, c_{k+1}, \dots, c_n\Big)$$

→ sum up the columns with the j^{th} column to cancel terms

$$= \mathrm{Det}\left(c_1, \dots, -(x^{-q}-1)c_k, \dots, c_{k-1}, c_{k+1}, \dots, c_n\right)$$

j^{th} column

$$= -(x^{-q}-1)\,c_k\,\mathrm{Det}(c_1, \dots, c_k, \dots, c_n)$$

$$= -(x^{-q}-1)\,\mathrm{Det}\left(M_k(0, c_j)\right).$$

Denote $\mathrm{Det}(M(a,b)) = \Delta_{a,b}$. Then

$$(x^{-q}-1)\,\Delta_{0,c_j} = \pm\,(x^{-p}-1)\,\Delta_{0,c_k}.$$

Since the indices of the regions are determined up to an additive constant, if c_ℓ and c_m are two more columns of M of index r and s, resp.ly then.

$$\underbrace{\begin{aligned} (x^{r-q}-1)\,\Delta_{\ell,j}(x) &= \pm\,(x^{r-r}-1)\,\Delta_{\ell,k}(x) \\ (x^{q-s}-1)\,\Delta_{k,\ell}(x) &= \pm\,(x^{q-r}-1)\,\Delta_{k,m}(x) \end{aligned}}_{\text{exercise!}}$$

$$(x^{r-q}-1)\,\Delta_{\ell,j}(x) = \pm\,(x^{r-p}-1)\,\Delta_{\ell,k}(x)$$

$$(x^{q-s}-1)\,\Delta_{k,\ell}(x) = \pm\,(x^{q-r}-1)\,\Delta_{k,m}(x)$$

$$\Rightarrow \quad \Delta_{\ell,k}(x) = \Delta_{k,\ell}(x) \underset{**}{=} \pm\,\frac{(x^{q-r}-1)\,\Delta_{k,m}(x)}{x^{q-s}-1}.$$

$$\Rightarrow \quad \Delta_{\ell,j}(x) = \pm\,\frac{(x^{r-p}-1)\,(x^{q-r}-1)\,\Delta_{k,m}(x)}{x^{q-s}-1}$$

Set $p = r+1 \;\wedge\; s = q+1$ → Two adjacent region indices differ by 1.

$$\Delta_{\ell,j}(x) = \pm\, x^{q-r}\,\Delta_{k,m}(x)\,, \quad q-r \in \mathbb{Z}.$$

Thus, whenever we remove two columns from the matrix of consecutive index, the determinants differ by $\pm x^{q-r}$. $\square$

Now, we check the effect of the Reidemeister moves to the Alexander matrix and its determinant.

definition: Two matrices M_1, M_2 are said to be ε-equivalent if it's possible to transform one another by a sequence of

α.) Multiplying a row or a column by $\underline{\underline{-1}}$

β.) Swapping the rows or columns

γ.) Adding one row or column to another

$\underline{\underline{\delta}}$.) Add / remove a row and a column where the "corner" entry is $\underline{\underline{1}}$, the rest is $\underline{\underline{0}}$:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \underset{\delta}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & c \\ 0 & d & e & f \\ 0 & g & h & i \end{pmatrix}$$

ε.) Multiply / divide a column by x.

Observation:

If $M_1 \underset{\epsilon}{\sim} M_2$ then

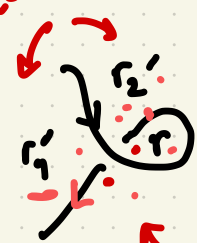
$$\det(M_1) = \pm x^k \det(M_2), \quad k \in \mathbb{Z}.$$

Claim: Equivalent knot diagrams have ϵ-equivalent Alexander matrices.

Proof:

K: 

$\longrightarrow$

K': 

adding 1 column, 1 row.

New matrix: delete

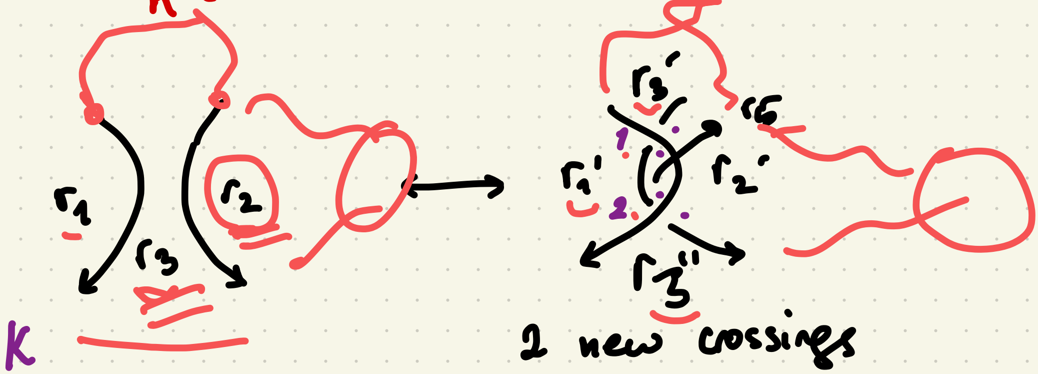
$$\mathcal{M}_{K'} = \begin{array}{c} \\ 1' \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} r & r_1' & r_2' & \cdots \end{array} \\ \left(\begin{array}{c|ccc} x & -1 & 1 & 0 \cdots 0 \\ \hline 0 & & & \\ \vdots & & M_K & \\ 0 & & & \end{array}\right) \end{array} \longrightarrow \begin{array}{c} \begin{array}{cc} r & \cdots \end{array} \\ \left(\begin{array}{c|c} \pm x & 0 \cdots 0 \\ \hline 0 & \\ \vdots & M_K(1,2) \\ 0 & \end{array}\right) \end{array}$$

$$\begin{array}{c} r \cdots \\ \begin{pmatrix} x & 0 \cdots 0 \\ 0 & \\ \vdots & M_{1,2} \\ 0 & \end{pmatrix} \end{array} \xrightarrow[\text{Multiply } r_1' \text{ by } -1]{\text{Divide } r_1 \text{ by } x} \begin{array}{c} r \\ \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & M_{1,2} \\ 0 & \end{array}\right) \end{array}$$

$$\Big\downarrow \text{ Remove border}$$

$$M_{1,2}$$

$\therefore$ $\Delta_K(x)$ is invariant under RI.



2 new crossings
2 new regions.

$$cr_1 = x r_5 - x r_2' - r_1' + r_3'$$

$$cr_2 = x r_2' - x r_5 - r_3'' + r_1'$$

$cr_1 = xr_5 - xr_2' - r_1' + r_3'$

$cr_2 = xr_2' - xr_5 - r_3'' + r_1'$

The resulting matrix:

$$\begin{array}{cccccccc} r_5 & r_2' & r_3'' & r_3' & r_1' & \cdot & \cdot & \cdot \\ \end{array}$$
$$\begin{bmatrix} x & -x & 0 & 1 & -1 & 0 & \cdots & 0 \\ -x & x & -1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & | & | & | & & & & \\ \vdots & u & v & w & & & M(2,3) & \\ 0 & | & | & | & & & & \end{bmatrix}$$

$\downarrow$ delete r_2' and r_3', divide c_1 by x

$$\begin{pmatrix} 1 & 0 & -1 & 0 & \cdots & 0 \\ -1 & -1 & 1 & 0 & \cdots & 0 \\ 0 & | & & & & \\ \vdots & v & & M_{2,3} & & \\ 0 & | & & & & \end{pmatrix} \xrightarrow{c_1 + c_3} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & 0 & \cdots & 0 \\ 0 & | & & & \\ \vdots & v & & M_{2,3} & \\ 0 & | & & & \end{pmatrix}$$

$\downarrow -c_2 + c_1$

$$\begin{pmatrix} -1 & 0 & \cdots & 0 \\ | & & & \\ v & & M_{2,3} & \\ | & & & \end{pmatrix} \xleftarrow{\text{remove } \partial} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & 0 \cdots & 0 \\ \vdots & | & & \\ \vdots & v & M_{2,3} & \\ & | & & \end{pmatrix}$$

Multiplying the first row by -1, we see that the resulting matrix after RII has the same determinant with the matrix representing the diagram before the move.

Exercise: Check the invariance under $\underline{\text{RIII}}$.

and adding up the resulting first row to the rows by multiplying it by "suitable" coef. we again obtain a "boundary" which can be deleted. Then we have

$$\det M_{\underline{K'}} = \det \underline{\begin{pmatrix} M_{\underline{2},\underline{3}} \end{pmatrix}}$$

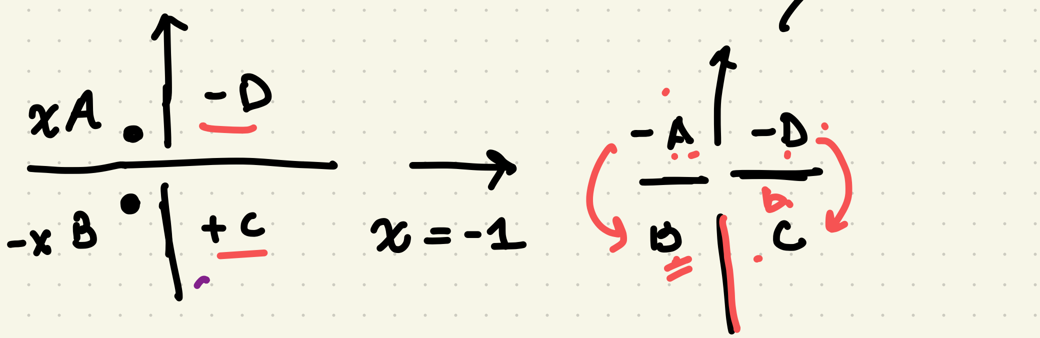
1) We have discussed

$$\Delta_K(x) = \Delta_{K^*}(x).$$

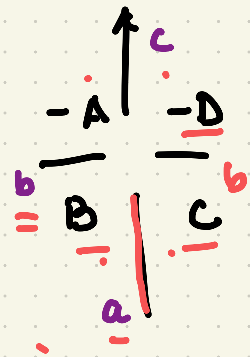
Exercise: Demonstrate a reasoning for why this equality holds.

2) How does the Alexander poly. behaves under reversion?

Relation with n-coloring:



are col.



Assume $A, B, C, D \in \mathbb{Z}_n$.

$b = -A + B = C - D$

$a = B + C$

$c = -A - D = -(A + D)$

$\Rightarrow \quad a + c = B + C - A - D$

$\quad = \underbrace{-A + B}_{b} + \underbrace{C - D}_{b}$

$\quad = 2b \pmod{n}$.

We find the coloring equation at this crossing by plugging $x = -1$.

definition : The absolute value of $\Delta_K(-1)$ is called the determinant of K.

$$\mathrm{Det}(K) = |\Delta_K(-1)|.$$

Substitute $x=-1$ in the Alexander matrix to find the coloring matrix.

Lecture 5 : March 6 2024

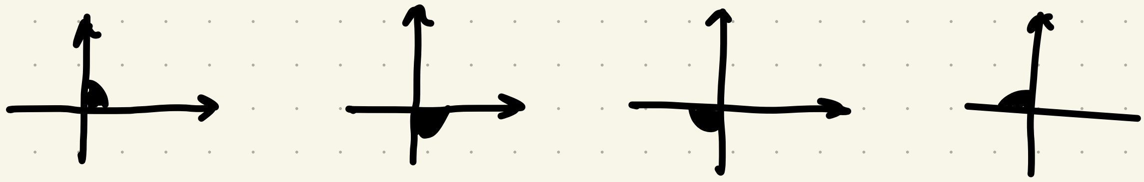
- State-sum formula for the Alexander Polynomial

Part I: Underlying combinatorics

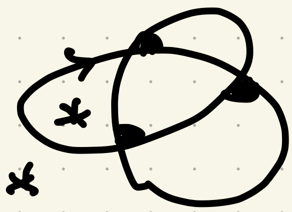
defn: A directed planar with 4 edges incident to each vertex is called a universe.

 is a trefoil universe.

defn: A state of a universe is an assignment of one marker per vertex in the forms:



so that each region in the graph receives no more than one marker.

Thus  is a state of the trefoil universe.

Two regions that receive stars are free of markers!

Proposition

A knot universe with n crossings (vertices) admits $n+2$ regions.

Proof: Every vertex is incident to 4 edges. Each edge is shared by two vertices.

Then, $\exists$ in total $2n$ edges in K.

Since K is a planar graph, by Euler's formula we have

$$n - 2n + f = 2 \;.$$

$$\Rightarrow f = n+2 \,.$$

Corollary: A state of a knot universe is a bijection from its regions without stars to its crossings.

$$s : \{\text{Regions without } *\} \to \{\text{Crossings}\}$$

Proposition: Every knot universe admits a
with a fixed choice of
adjacent stars on it
state.

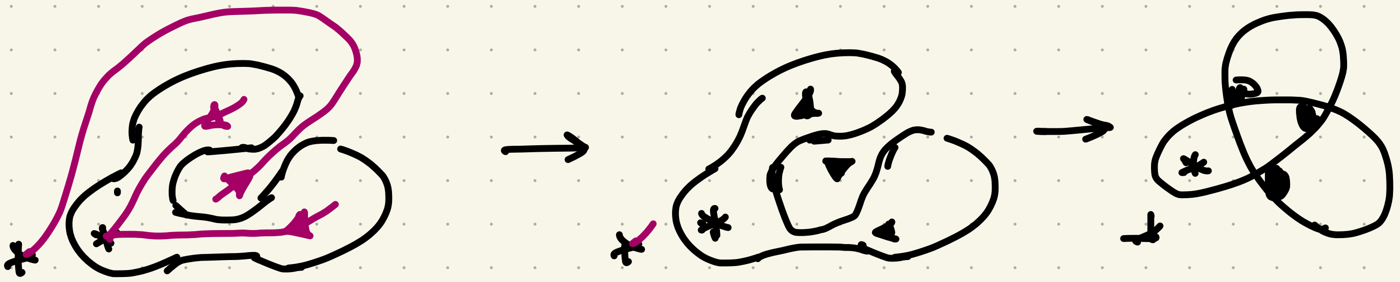
Proof: A circuit of a planar graph is a closed path that traverses each edge exactly once. Let us call a circuit that doesn't contain any vertices a Jordan trail. (can be considered as a simple closed curve in $\mathbb{R}^2$.)

Since K is a 4-regular graph, it admits Jordan trails. In fact, Jordan trails of K are in 1-1 correspondence with states of K:

By splitting all crossings in a state, one Jordan trail automatically appears.

Conversely, a choice of stars at the Jordan trail determines a specific state:

Grow two directed trees, each rooted at one star:



a Jordan trail on K

a state of K

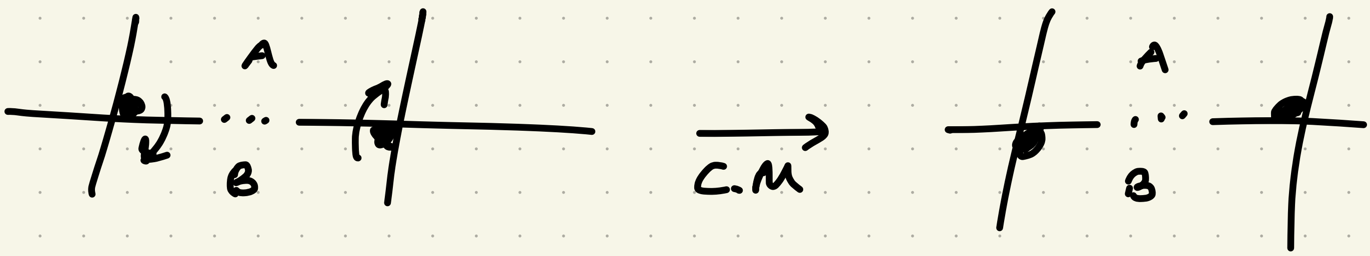
∎

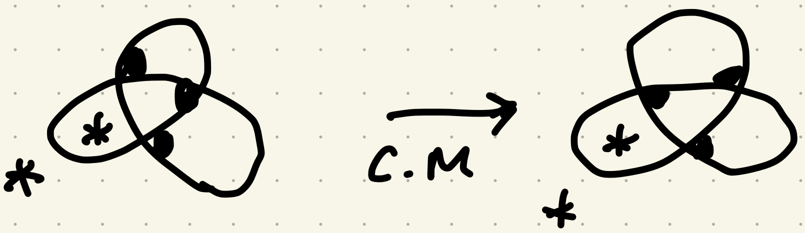
One can consider of states as region-vertex assignments, hence as permutations of the vertices relative to an ordering of the regions.

Hence, natural to think transposition of permutations.

Geometric interpretation of a transposition:

Clock moves : A state transposition!



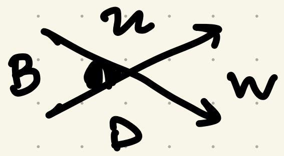


defn : A state is called a clocked state if it admit only clockwise transpositions and counterclocked if it admits only counterclockwise transpositions.

Thm : (Clock Theorem) [FKT, Kauffman, '83]
Let K be a knot universe and $\mathcal{S}$ be the set of states of K for a given choice of adjacent fixed stars.
Then $\mathcal{S}$ has a unique clocked state and a unique counter-clocked state.
Any state in $\mathcal{S}$ can be reached from the clocked state by a series of clockwise moves.
Hence any two states in $\mathcal{S}$ are connected by a series of state transpositions.

- By defining $S < S'$ whenever $\exists$ a series of clock moves connecting S' to S, $\mathcal{S}$ becomes a lattice.

Clarify state markers:



defn : Let S be a state of K. Define the sign of S by

$$\sigma(S) = (-1)^{b(S)}$$

where $b(S) :=$ # of black holes in S.

Recall that : Let $p \in S_n$, and let sign of p is denoted by $\mathrm{sgn}(p)$. Then,

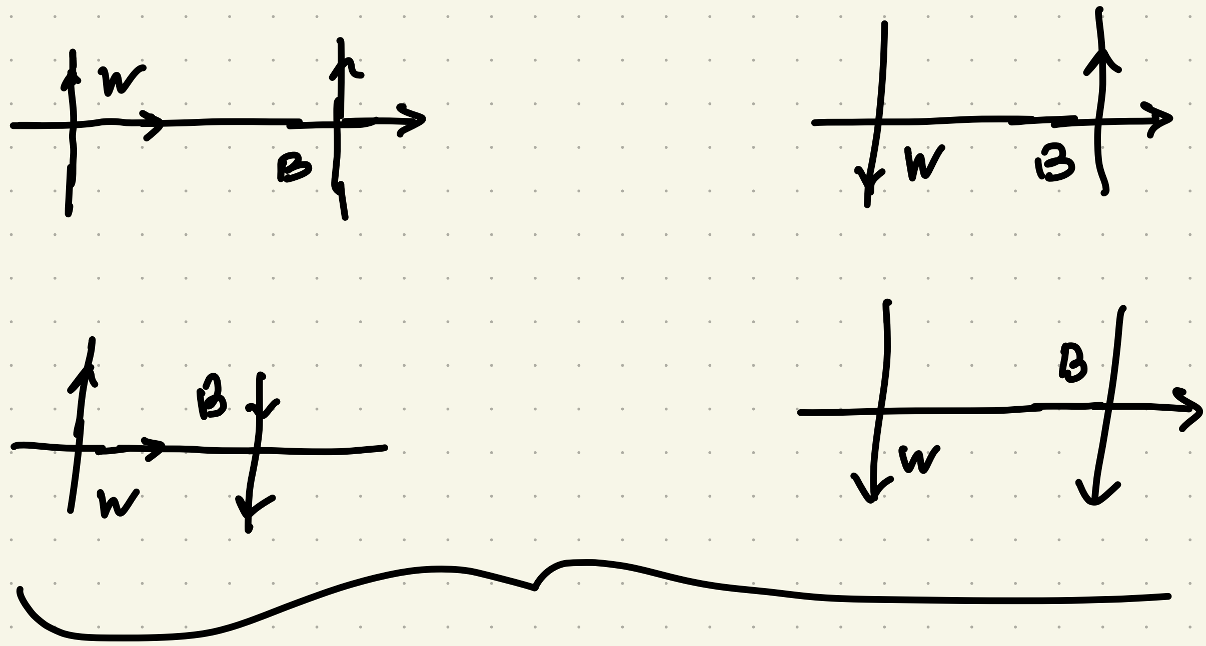
$\mathrm{sgn}(p) = (-1)^t$ where t is the # of transpositions to turn p into the identity permutation: $(1) \dots (n)$.

Each state corresponds to a permutation!

lemma : Let S and S' be states of an oriented universe so that S' is obtained from S by one transposition. Then

$$\mathrm{sgn}(S') = -\mathrm{sgn}(S).$$

Proof :



$\Rightarrow \quad \sigma(s') = \sigma(s).$

Proposition : $\mathcal{S}$, collection of states of K.

$P : \mathcal{S} \to S_n$, a permutation assignment for $\mathcal{S}$. Then, the signs of the states agree with the signs of their corresponding permutations.

i.e

$$\sigma(S) = \operatorname{sgn}(P(S)).$$

Proof : Choose the ordering of vertices and regions of K so that the region-vertex assignment

$$R_i \to V_i \qquad i = 1, \cdots, n$$

corresponds to a state S_0 and so that

$$\sigma(S_0) = 1.$$

Clearly $\mathcal{S}_0$ corresponds to the id permutation and so,

$$\sigma(S_0) = \operatorname{sgn}(P(S_0)) = 1.$$

By the clock theorem any other state S in $\mathcal{S}$ can be reached from $\mathcal{S}_0$ by a sequence of clock moves (or state transpositions). Let t be the number of such clock moves. Then by the previous lemma

$$\sigma(S) = (-1)^t$$

and since state transpositions $\xrightarrow[P]{}$ perm. transp.

$$\operatorname{sgn}(P(S)) = (-1)^t.$$

$$\therefore \quad \sigma(S) = \operatorname{sgn}(P(S)).$$

• — End of the 1st lecture — •

Part II : forming the state-sum.

Now, consider an oriented knot diagram K.

Consider the weights :

Remember, for an $n \times n$ square matrix M

$$\operatorname{Det} M = \sum_{\rho \in S_n} \operatorname{sgn}(\rho) \prod_{i=1}^{n} a_{i\rho(i)} .$$

I^{st} observation :

Let M is the Alexander matrix of K with n crossings and two adj. regions endowed with stars.

Let s be a state of the universe of K and p be the corresponding permutation.

Observe :

$$\prod_{i=1}^{n} a_{ip(i)} = \prod_{i=1}^{n} \text{weight of the state marker in } s \text{ at the crossing } c_i$$

$= \langle K | s \rangle$ ← denote this product by this

And we already know

$$\operatorname{sgn}(p) = \sigma(s)$$

Corollary: $\boxed{\Delta_K(x) \doteq \sum_{s \in \{\text{States}\}} \sigma(s) \langle K | s \rangle .}$

Day 4: Mock Alexander Polynomials

QI: Can we change the labelling and get rid of the sign calculation?

Yes.

QII: Can we generalize the state-sum polynomial to other knotted objects?

Yes.

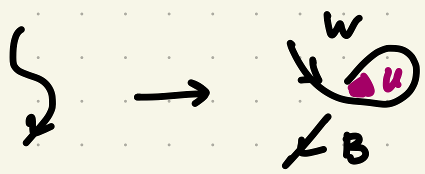
Part I: Changing the labeling

Let

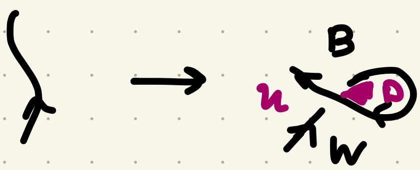
Claim: To get an invariant $WB = 1$ and $u = D = 1$.

Proof: Check the behavior under Reidemeister moves.

For ex:



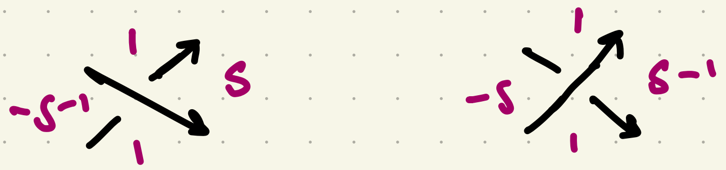
$u = 1$



$D = 1$

$BW = 1$

Changing the labeling so that signs are absorbed:



How to generalize the state-sum to other knotted objects:

Some Results from Euler's formula

Prop: Let $\mathcal{U}$ be the universe of a connected diagram with n crossings, tightly embedded in a surface of genus g.

We have:

i-) If $\mathcal{U}$ is the universe of a link diagram, then $f - n = 2 - 2g$.

Pf: n 4-valent vertices $\Rightarrow$ $2n$ edges

Euler's formula

$$n - 2n + f = 2 - 2g \iff f - n = 2 - 2g.$$

Cor: If a link diagram is embedded in T^2

$\Rightarrow$ $f = n$.

ii.) If $\mathcal{U}$ is the universe of a linkoid diagram with m knotoid components even,

$$f - n = 2 - 2g - m.$$

Proof: $\underset{\substack{\Downarrow \\ \text{1-valent}}}{n + 2m}$ vertices $\Rightarrow$ $2n + m$ edges.

$$\underset{\text{E.F}}{\Rightarrow} \quad (n+2m) - (2n+m) + f = 2 - 2g.$$

$$\Leftrightarrow \quad -n + m + f = 2 - 2g.$$

$$\Rightarrow \quad f - n = 2 - 2g - m.$$

Corollary: If $g = 0$ and $m = 2$ then $f = n$.

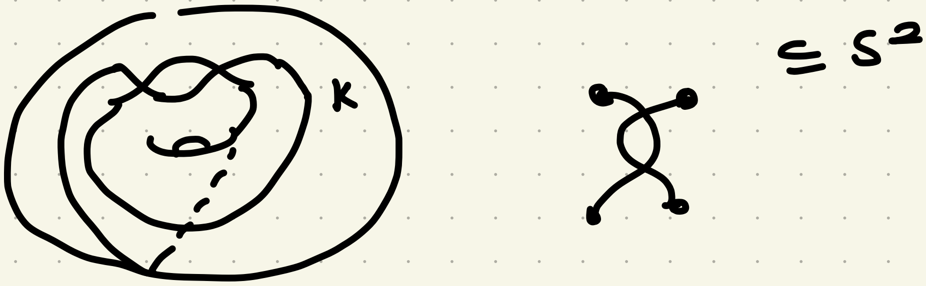
Corollary: If $g \geq 1$ then $n > f$ $\forall\, m \geq 1$.

defn: If L has the property

$$f = n$$

where $n = \#$ of 4-valent vertices and
$f = \#$ of faces of the universe of L

then L is called admissable.



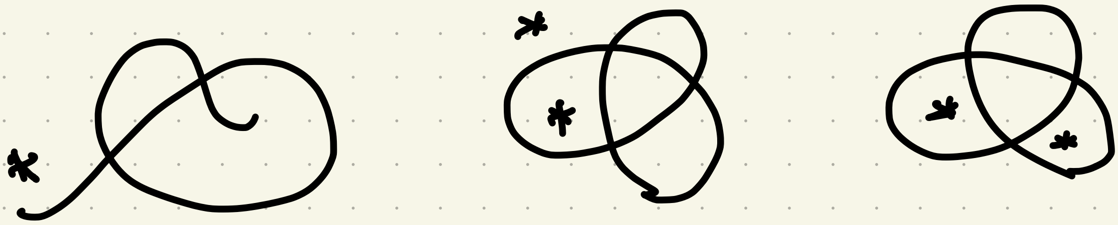
Two admissable diagram

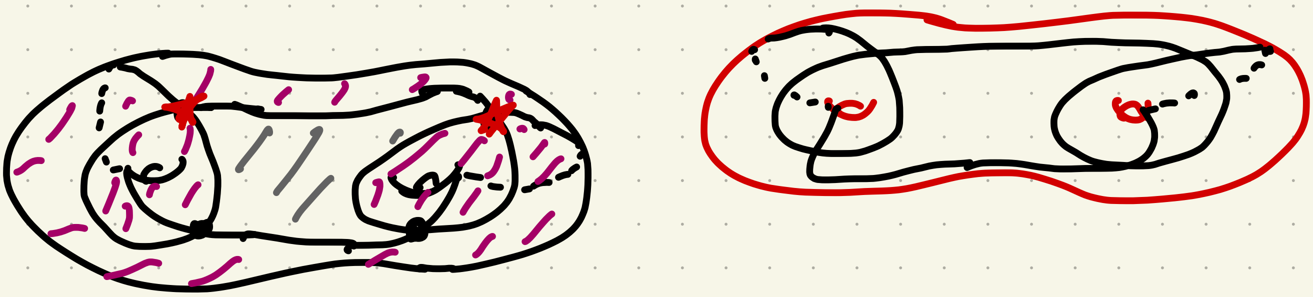
What if $\mathcal{f}$ is non-admissable?

Purpose: To define the state-sum on any connected diagram.

<u>defn:</u> A starred link or linkoid diagram is a link/linkoid diagram that is endowed with stars at its regions or crossings so that $f = n$ is satisfied for regions and crossings without stars.

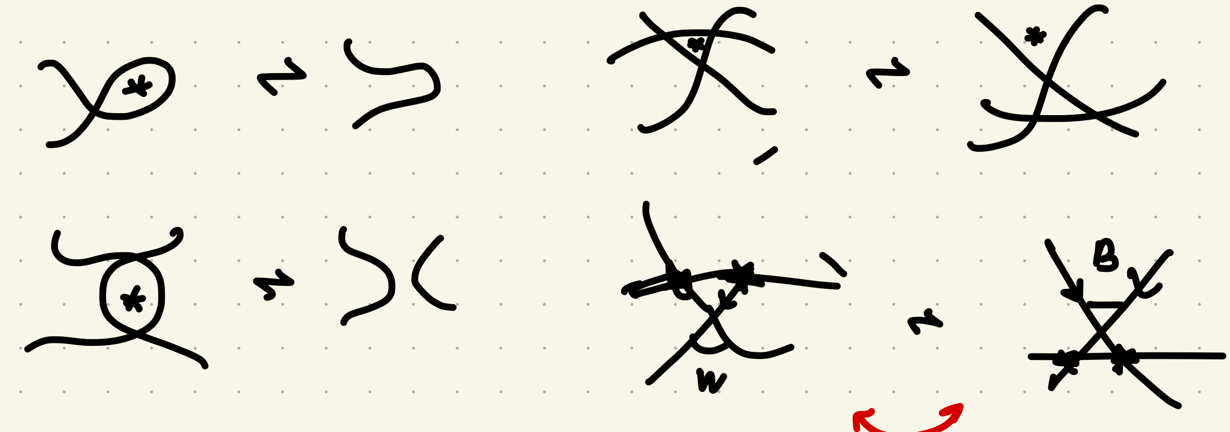
ex



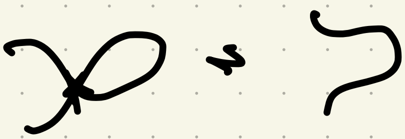


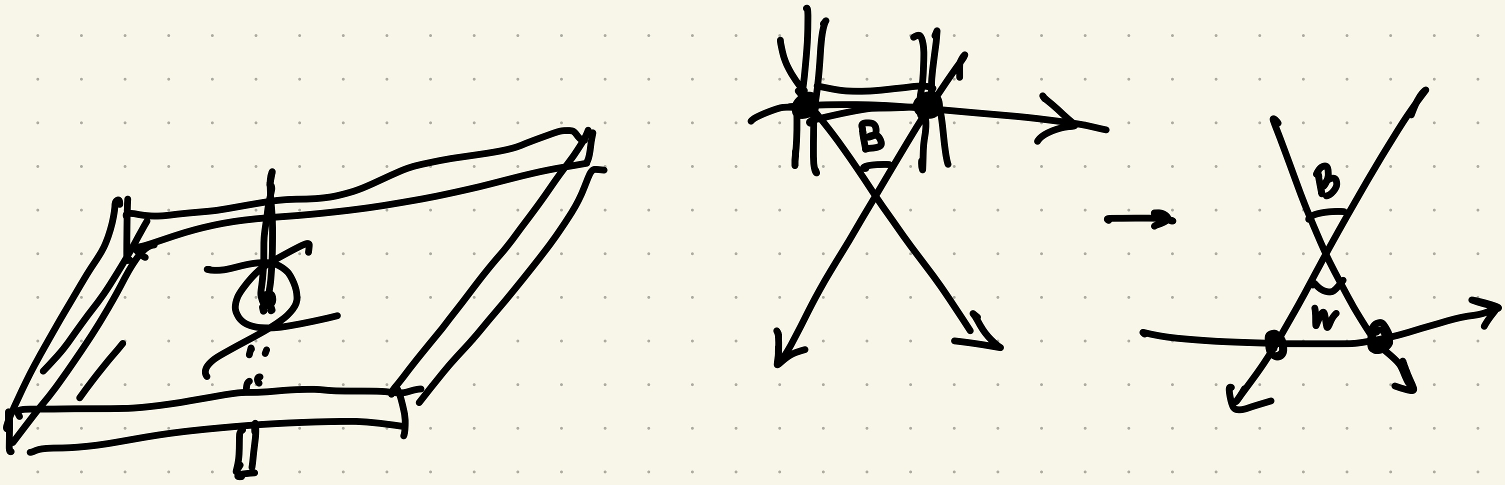
4 vertices
2 faces $\Rightarrow$ Star 2 crossings

Starred diagrams are considered up to star-equivalence, so the following moves are not allowed:



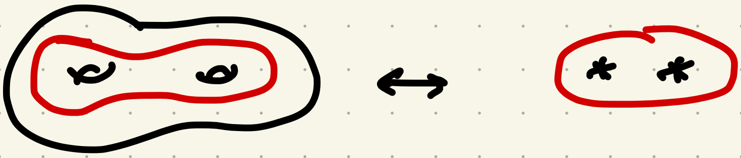
why don't we allow this?





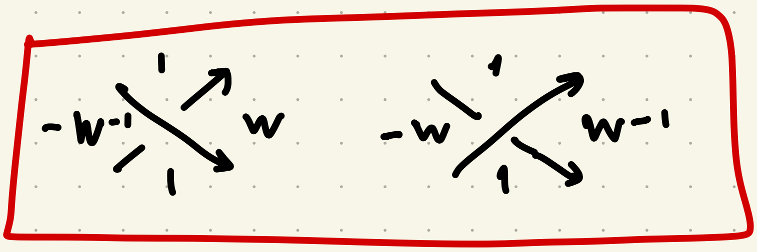
NOTE: Starred knots can be considered as knots in handlebodies.

But for 1-1 correspondence, need to generalize the concept of starred knotoid:



def: Let K be a starred link / linkoid diagram. Define

$$\nabla_K(W) = \sum_{s \in \{\text{states of } K\}} \langle K \mid s \rangle .$$



$\nabla_K(W)$ is called the Mock Alexander polynomial of K.

Thm: $\nabla_K(W)$ is an invariant of starred links and linkoids.

Pf: exercise. :)

note: If $\mathcal{L}$ is a connected link diagram with two of its adjacent regions starred, then

$\nabla_K(W)$ gives the Conway-Alexander polynomial.