# Introduction to singular perturbations and boundary layers

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CIMPA-ICTP Fellowship “Research in Pairs” 2024

June 17–20, 2024
CIRM

# Overview

1. Singular perturbations: examples and concepts
2. Singular perturbations: asymptotic expansions
3. Penalization method for Dirichlet boundary conditions

   $\longrightarrow$ Blackboard
4. Penalization method for Neumann or Robin boundary conditions

   $\longrightarrow$ Notes

# Introduction

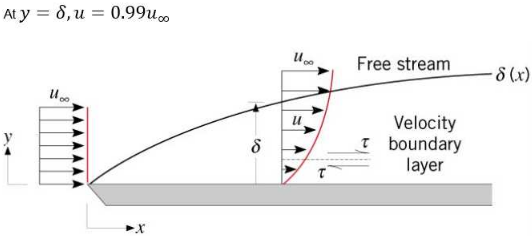


Figure: Velocity boundary layer in fluid mechanics.[1]

[1] https://help.altair.com/hwcfdsolvers/acusolve

# Introduction

Other examples of singularly perturbed problems ($\varepsilon > 0$ is a small parameter):

- Convection-diffusion-reaction problem

$$\begin{cases} -\varepsilon \Delta u_\varepsilon + b(x, u_\varepsilon) \cdot \nabla u_\varepsilon + f(x, u_\varepsilon) = 0, \quad x \in \Omega \\ u_\varepsilon(x) = 0, \quad x \in \partial\Omega. \end{cases}$$

- Problem of a thin beam

$$\begin{cases} \varepsilon u_\varepsilon^{(4)} - u_\varepsilon'' = \lambda^2 u_\varepsilon, \quad 0 < x < 1 \\ u_\varepsilon(0) = u_\varepsilon(1) = u_\varepsilon'(0) = u_\varepsilon'(1) = 0. \end{cases}$$

# Overview

# Singular perturbations

We consider a family of perturbation problems with a small parameter $\varepsilon$:

$$P_\varepsilon(x, u^\varepsilon, \partial_x u^\varepsilon, \cdots; \varepsilon) = 0.$$

Formally, when $\varepsilon \to 0$, we have the following limit problem:

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## Definition (Singular perturbation)

Let $P$ be a differential operator in $\mathbb{R}^n$, $n \geq 1$, in the form,

$$P(\mathbf{x}, D) = \sum_{|\alpha| \leq m} a_\alpha(\mathbf{x}) D^\alpha, \quad a_\alpha \neq 0 \text{ for some } |\alpha| = m,$$

where we use multi-index notation. $\deg(P) = m$.

- $\deg(P_0) < \deg(P_\varepsilon)$: singular perturbation problems
- $\deg(P_0) = \deg(P_\varepsilon)$: regular perturbation problems

# Examples

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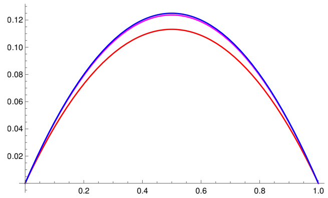

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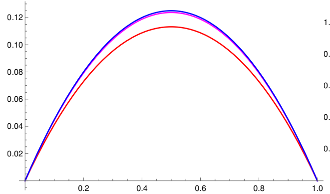


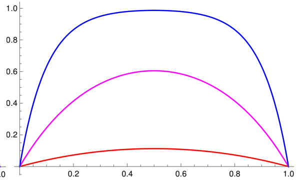

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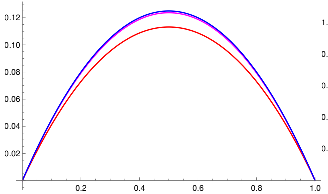


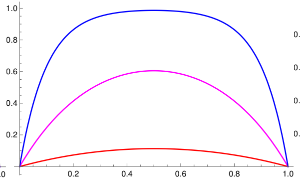


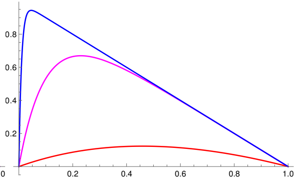

# Reaction-diffusion equations in 1D

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Variational formulation:

$$\begin{cases} \text{Find } u_\varepsilon \in H_0^1(0,1) \text{ such that } \forall v \in H_0^1(0,1), \\ \varepsilon \int_0^1 u_\varepsilon' v' + \int_0^1 u_\varepsilon v = \int_0^1 fv. \end{cases} \tag{2}$$

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There exists a unique solution $u_\varepsilon \in H_0^1(0,1)$ of (2) using the Lax-Milgram theorem. We are interested in the limit when $\varepsilon \to 0$.

**Theorem**

*The solution $u_\varepsilon$ of* (1) *converges in* $L^2(0,1)$, *when* $\varepsilon \to 0$, *to* $u_0 = f$.

**Proof.**

By taking $v = u_\varepsilon$ in (2) and using Cauchy-Schwarz, we obtain

$$\varepsilon \|u_\varepsilon'\|^2_{L^2(0,1)} + \|u_\varepsilon\|^2_{L^2(0,1)} = \int_0^1 f u_\varepsilon \leq \|f\|_{L^2(0,1)} \|u_\varepsilon\|_{L^2(0,1)} \leq \frac{1}{2}\|f\|^2_{L^2(0,1)} + \frac{1}{2}\|u_\varepsilon\|^2_{L^2(0,1)}.$$

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It follows that $\sqrt{\varepsilon} u_\varepsilon'$ and $u_\varepsilon$ are bounded in $L^2(0,1)$ independently of $\varepsilon$. There exists a subsequence $\varepsilon' \to 0$ and $u_0 \in L^2(0,1)$ such that

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We can show that the weak convergence is valid for the whole sequence $\varepsilon \to 0$. It follows

$$\begin{aligned} \varepsilon \left\|u_\varepsilon'\right\|^2_{L^2(0,1)} + \|u_\varepsilon - u_0\|^2_{L^2(0,1)} &= (f, u_\varepsilon) - 2\,(u_\varepsilon, u_0) + \|u_0\|^2_{L^2(0,1)} \\ &\to (f, u_0) - \|u_0\|^2_{L^2(0,1)} = 0, \end{aligned}$$

hence the strong convergence in $L^2(0,1)$. □

# Reaction-diffusion equations in 1D

## Thickness of the boundary layer, approximate solution, approximate correctors

### Remark

The limit solution does not satisfy the same boundary conditions as $u_\varepsilon$ if $f(0) \neq 0$ or $f(1) \neq 0$. For instance, if $f \in H^1(0,1)$ but not in $H^1_0(0,1)$ then $u_\varepsilon(0) = 0 \nrightarrow u_0(0)$ or $u_\varepsilon(1) = 0 \nrightarrow u_0(1)$. Since $H^1(0,1) \hookrightarrow C^0([0,1])$, $u_\varepsilon \nrightarrow u_0$ in $H^1(0,1)$.
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An approximation of the solution is given by

$$u_\varepsilon(x) \approx u_0(x) + \theta^l\left(\frac{x}{\eta_0}\right) + \theta^r\left(\frac{1-x}{\eta_1}\right) \tag{3}$$

where $0 \leq \eta_0, \eta_1 \ll 1$, $\theta^{l,r}(\xi) \underset{\xi\to\infty}{\rightarrow} 0$ and $\frac{d^k}{d\xi^k}\theta^{l,r}(\xi) \underset{\xi\to\infty}{\rightarrow} 0$.

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$$-\frac{\varepsilon}{\eta_1^2}(\theta^r)''\left(\frac{1-x}{\eta_1}\right) + \theta^r\left(\frac{1-x}{\eta_1}\right) = 0$$

Interesting choice: $\eta_1 = \sqrt{\varepsilon}$. Then $\theta^r(\xi) = B_+ e^{\xi} + B_- e^{-\xi} = B_- e^{-\xi}$.
From $u_\varepsilon(1) = 0$, we get $B_- = -u_0(1)$.

An approximation of the solution is given by

$$u_\varepsilon(x) \approx u_0(x) + \theta^l\left(\frac{x}{\sqrt{\varepsilon}}\right) + \theta^r\left(\frac{1-x}{\sqrt{\varepsilon}}\right) \tag{5}$$

where the approximate correctors near $x = 0$ and $x = 1$ are given by

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$$\theta_\varepsilon(x) = \theta^l\left(\frac{x}{\sqrt{\varepsilon}}\right)\sigma(x) + \theta^r\left(\frac{1-x}{\sqrt{\varepsilon}}\right)\sigma(1-x). \tag{7}$$

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Since $\sigma' = \sigma'' = 0$ near $x = 0$ and $x = 1$ and $\dfrac{e^{-\frac{\delta}{\sqrt{\varepsilon}}}}{\varepsilon^n} \to 0$ for $\delta > 0$, we obtain

$$\|R_\varepsilon\|_{L^2(0,1)} \leq C\varepsilon^n.$$

We deduce

$$\begin{cases} -\varepsilon w_\varepsilon'' + w_\varepsilon = \varepsilon u_0'' - R_\varepsilon = \varepsilon f'' - R_\varepsilon, & 0 < x < 1, \\ w_\varepsilon = 0, & \text{at } x = 0, 1 \end{cases} \tag{9}$$

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Multiplying by $w_\varepsilon$ and integrating by parts, we obtain (assuming $f'' \in L^2$)

$$\begin{aligned} \varepsilon \|w_\varepsilon'\|^2_{L^2(0,1)} + \|w_\varepsilon\|^2_{L^2(0,1)} &\le \varepsilon \|f''\|_{L^2(0,1)} \|w_\varepsilon\|_{L^2(0,1)} + \|R_\varepsilon\|_{L^2(0,1)} \|w_\varepsilon\|_{L^2(0,1)} \\ &\le C\varepsilon \|w_\varepsilon\|_{L^2(0,1)} \\ &\le \frac{C^2}{2}\varepsilon^2 + \frac{1}{2}\|w_\varepsilon\|^2_{L^2(0,1)} \end{aligned}$$

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It follows, since $w_\varepsilon \in H^1_0(0,1)$, that

$$\|w_\varepsilon\|_{L^2(0,1)} \leq C\varepsilon.$$
$$\|w_\varepsilon\|_{H^1(0,1)} \leq C\sqrt{\varepsilon}.$$

# Reaction-diffusion equations in 1D

## Theorem

*Assume that* $f \in H^2(0,1)$*. Let* $u_\varepsilon, u_0$ *be the solutions of*

$$\begin{cases} -\varepsilon u_\varepsilon'' + u_\varepsilon = f, \quad 0 < x < 1, \\ u_\varepsilon(0) = u_\varepsilon(1) = 0, \end{cases} \tag{10}$$

*and*

$$\begin{cases} u_\varepsilon = f, \quad 0 < x < 1, \\ \text{without boundary conditions}, \end{cases} \tag{11}$$

*respectively and* $\theta_\varepsilon$ *be defined in* (7)*. Then there exists a constant* $C > 0$ *independent of* $\varepsilon$ *such that*

$$\|u_\varepsilon - (u_0 + \theta_\varepsilon)\|_{L^2(0,1)} \leq C\varepsilon,$$

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## Remark

We can show the same result for the approximation using approximate correctors, i.e. replacing $\theta_\varepsilon$ by $\theta^l\left(\frac{x}{\sqrt{\varepsilon}}\right) + \theta^r\left(\frac{1-x}{\sqrt{\varepsilon}}\right)$.

# Convection-diffusion equations in 1D

Let $f \in L^2(0,1)$, we consider the following model problem:

$$\begin{cases} -\varepsilon u_\varepsilon'' - u_\varepsilon' = f, \quad 0 < x < 1, \\ u_\varepsilon(0) = u_\varepsilon(1) = 0. \end{cases} \tag{12}$$

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Variational formulation:

$$\begin{cases} \text{Find } u_\varepsilon \in H_0^1(0,1) \text{ such that } \forall v \in H_0^1(0,1), \\ \varepsilon \int_0^1 u_\varepsilon' v' - \int_0^1 u_\varepsilon' v = \int_0^1 fv. \end{cases} \tag{13}$$

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There exists a unique solution $u_\varepsilon \in H_0^1(0,1)$ of (13) using the Lax-Milgram theorem. We are interested in the limit when $\varepsilon \to 0$.

Formally, the limit problem is obtained by setting $\varepsilon = 0$:

$$-u_0' = f, \quad 0 < x < 1. \tag{14}$$

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Since it is a first-order PDE (transport equation), we need to impose one boundary condition, but on which boundary?

# Convection-diffusion equations in 1D

Let us assume the following ansatz for the approximation:

$$u_\varepsilon(x) \approx u_0(x) + \theta^l\left(\frac{x}{\eta_0}\right) + \theta^r\left(\frac{1-x}{\eta_1}\right) \tag{15}$$

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- Near $x = 0$, we neglect $\theta^r$, thus the equation becomes

$$-\varepsilon u_0''(x) - \frac{\varepsilon}{\eta_0^2}(\theta^l)''\left(\frac{x}{\eta_0}\right) - u_0'(x) - \frac{1}{\eta_0}(\theta^l)'\left(\frac{x}{\eta_0}\right) \approx f$$

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  Interesting choice : $\eta_0 = \varepsilon$. Then $\theta^l(\xi) = A_0 + A_- e^{-\xi} = A_- e^{-\xi}$.
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- Near $x = 1$, we neglect $\theta^l$. Similarly, we get

$$-\frac{\varepsilon}{\eta_1^2}(\theta^r)''\left(\frac{1-x}{\eta_1}\right) + \frac{1}{\eta_1}(\theta^r)'\left(\frac{1-x}{\eta_1}\right) = 0$$

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  Interesting choice : $\eta_1 = \varepsilon$. Then $\theta^r(\xi) = B_0 + B_+ e^{\xi} = 0$.
  $\rightarrow$ No boundary layer at $x = 1$.

# Convection-diffusion equations in 1D

The limit problem is

$$\begin{cases} -u_0' = f, & 0 < x < 1 \\ u_0(1) = 0, & \text{at } x = 1. \end{cases} \tag{16}$$

# Convection-diffusion equations in 1D

An approximation of the solution is given by

$$u_\varepsilon(x) \approx u_0(x) + \theta\left(\frac{x}{\varepsilon}\right) \tag{17}$$

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$$\|R_\varepsilon\|_{L^2(0,1)} \leq C\varepsilon^n.$$

We deduce

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Since multiplying by $w_\varepsilon$ and integrating by parts removes the term $\int_0^1 w_\varepsilon' w_\varepsilon = 0$, we multiply by $e^x w_\varepsilon$ instead, we obtain (assuming $f' \in L^2$)

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$$\varepsilon \int_0^1 w_\varepsilon' (w_\varepsilon e^x)' + \int_0^1 \frac{w_\varepsilon^2}{2} e^x = \dots$$

$$\varepsilon \int_0^1 (w_\varepsilon')^2 e^x + \varepsilon \int_0^1 w_\varepsilon' w_\varepsilon e^x + \int_0^1 \frac{w_\varepsilon^2}{2} e^x = \dots$$

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hence

$$\varepsilon \|w_\varepsilon'\|_{L^2(0,1)}^2 + \frac{1-2\varepsilon}{4} \|w_\varepsilon\|_{L^2(0,1)}^2 \leq C\varepsilon^2.$$

We deduce

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Since multiplying by $w_\varepsilon$ and integrating by parts removes the term $\int_0^1 w_\varepsilon' w_\varepsilon = 0$, we multiply by $e^x w_\varepsilon$ instead, we obtain (assuming $f' \in L^2$)

$$-\varepsilon \int_0^1 w_\varepsilon'' w_\varepsilon e^x - \int_0^1 w_\varepsilon' w_\varepsilon e^x = -\varepsilon \int_0^1 f' w_\varepsilon e^x - \int_0^1 R_\varepsilon w_\varepsilon e^x$$

$$\varepsilon \int_0^1 w_\varepsilon' (w_\varepsilon e^x)' + \int_0^1 \frac{w_\varepsilon^2}{2} e^x = \dots$$

$$\varepsilon \int_0^1 (w_\varepsilon')^2 e^x + \varepsilon \int_0^1 w_\varepsilon' w_\varepsilon e^x + \int_0^1 \frac{w_\varepsilon^2}{2} e^x = \dots$$

$$\varepsilon \|w_\varepsilon'\|^2_{L^2(0,1)} + \frac{1-\varepsilon}{2} \|w_\varepsilon\|^2_{L^2(0,1)} \le \varepsilon \|f'\|_{L^2(0,1)} \|w_\varepsilon e^x\|_{L^2(0,1)} + \|R_\varepsilon\|_{L^2(0,1)} \|w_\varepsilon e^x\|_{L^2(0,1)}$$

$$\varepsilon \|w_\varepsilon'\|^2_{L^2(0,1)} + \frac{1-\varepsilon}{2} \|w_\varepsilon\|^2_{L^2(0,1)} \le C\varepsilon \|w_\varepsilon\|_{L^2(0,1)}$$

$$\varepsilon \|w_\varepsilon'\|^2_{L^2(0,1)} + \frac{1-\varepsilon}{2} \|w_\varepsilon\|^2_{L^2(0,1)} \le C^2\varepsilon^2 + \frac{1}{4} \|w_\varepsilon\|^2_{L^2(0,1)}$$

hence

$$\varepsilon \|w_\varepsilon'\|^2_{L^2(0,1)} + \frac{1-2\varepsilon}{4} \|w_\varepsilon\|^2_{L^2(0,1)} \le C\varepsilon^2.$$

It follows, since $w_\varepsilon \in H_0^1(0,1)$ and $\varepsilon \ll 1$, that

$$\|w_\varepsilon\|_{L^2(0,1)} \le C\varepsilon.$$
$$\|w_\varepsilon\|_{H^1(0,1)} \le C\sqrt{\varepsilon}.$$

## Theorem

*Assume that $f \in H^1(0,1)$. Let $u_\varepsilon, u_0$ be the solutions of*

$$\begin{cases} -\varepsilon u_\varepsilon'' - u_\varepsilon' = f, \quad 0 < x < 1, \\ u_\varepsilon(0) = u_\varepsilon(1) = 0, \end{cases} \tag{22}$$

*and*

$$\begin{cases} -u_0' = f, \quad 0 < x < 1, \\ u_0(1) = 0, \end{cases} \tag{23}$$

*respectively and $\theta_\varepsilon$ be defined in (19). Then there exists a constant $C > 0$ independent of $\varepsilon$ such that*

$$\|u_\varepsilon - (u_0 + \theta_\varepsilon)\|_{L^2(0,1)} \le C\varepsilon,$$

$$\|u_\varepsilon - (u_0 + \theta_\varepsilon)\|_{H^1(0,1)} \le C\sqrt{\varepsilon}.$$

# Convection-diffusion equations in 1D

## Theorem

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## Remark

We can show the same result for the approximation using the approximate corrector, i.e. replacing $\theta_\varepsilon$ by $\theta\left(\frac{x}{\varepsilon}\right)$.

# Other model problems in 1D

Let $f \in L^2(0,1)$, we consider the following model problem:[2]

$$\begin{cases} -\varepsilon u_\varepsilon^{(4)} + u_\varepsilon' = f, \quad 0 < x < 1, \\ u_\varepsilon(0) = u_\varepsilon'(0) = u_\varepsilon(1) = u_\varepsilon'(1) = 0. \end{cases} \tag{24}$$

[2] Couches limites en Océanographie - Anne-Laure Dalibard - Une question, un chercheur (2019) : https://www.carmin.tv/fr/speakers/anne-laure-dalibard

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Variational formulation:

$$\begin{cases} \text{Find } u_\varepsilon \in H_0^2(0,1) \text{ such that } \forall v \in H_0^2(0,1), \\ -\varepsilon \int_0^1 u_\varepsilon'' v'' + \int_0^1 u_\varepsilon' v = \int_0^1 f v. \end{cases} \tag{25}$$

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There exists a unique solution $u_\varepsilon \in H_0^2(0,1)$ of (25) using the Lax-Milgram theorem. We are interested in the limit when $\varepsilon \to 0$.

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## Exercice

Establish the limit problem and a convergence result using the boundary layer approach.

[2] Couches limites en Océanographie - Anne-Laure Dalibard - Une question, un chercheur (2019) : https://www.carmin.tv/fr/speakers/anne-laure-dalibard

# Overview

An approximation of the solution was given by

$$u_\varepsilon(x) \approx u_0(x) + \theta_\varepsilon(x) = u_0(x) + \theta^l\left(\frac{x}{\sqrt{\varepsilon}}\right)\sigma(x) + \theta^r\left(\frac{1-x}{\sqrt{\varepsilon}}\right)\sigma(1-x) \qquad (26)$$

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- $u_0 + \theta_\varepsilon$: zeroth order approximation at order $\varepsilon^0$

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- $u_0$: zeroth order outer solution
- $\theta_\varepsilon$: zeroth order inner solution

We generalize the asymptotic expansions in powers of $\varepsilon$:

$$u_\varepsilon(x) \approx \sum_{j=0}^{\infty} \varepsilon^j (u_j + \theta_{j,\varepsilon}) \approx \sum_{j=0}^{\infty} \varepsilon^j \Big(u_j(x) + \theta_j^l\Big(\frac{x}{\sqrt{\varepsilon}}\Big) + \theta_j^r\Big(\frac{1-x}{\sqrt{\varepsilon}}\Big)\Big) \tag{27}$$

where $\theta_j^{l,r}(\xi) \underset{\xi\to\infty}{\to} 0$ and $\frac{d^k}{d\xi^k}\theta_j^{l,r}(\xi) \underset{\xi\to\infty}{\to} 0$.

We substitute in the reaction-diffusion equation and identify each power of $\varepsilon$:

$$-\varepsilon\Big(\sum_{j=0}^{\infty}\varepsilon^j\Big(u_j''(x)+\frac{1}{\varepsilon}(\theta_j^l)''\Big(\frac{x}{\sqrt{\varepsilon}}\Big)+\frac{1}{\varepsilon}(\theta_j^r)''\Big(\frac{1-x}{\sqrt{\varepsilon}}\Big)\Big)\Big)+\sum_{j=0}^{\infty}\varepsilon^j\Big(u_j(x)+\theta_j^l\Big(\frac{x}{\sqrt{\varepsilon}}\Big)+\theta_j^r\Big(\frac{1-x}{\sqrt{\varepsilon}}\Big)\Big)=f$$

# Asymptotic expansions

## Reaction-diffusion equation

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$$\begin{cases}-(\theta_0^l)''+\theta_0^l=0\\ \theta_0^l(0)=-u_0(0)\end{cases}\quad\begin{cases}-(\theta_0^r)''+\theta_0^r=0\\ \theta_0^r(0)=-u_0(1)\end{cases}$$

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$$u_j=u_{j-1}''=f^{(2j)}$$

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We obtain $\theta_j^l(\xi_l)=-u_j(0)e^{-\xi_l}$ and $\theta_j^r(\xi_r)=-u_j(1)e^{-\xi_r}$.

We localize the approximate correctors and define the correctors

$$\theta_{j,\varepsilon}(x) = \theta_j^l\Big(\frac{x}{\sqrt{\varepsilon}}\Big)\sigma(x) + \theta_j^r\Big(\frac{1-x}{\sqrt{\varepsilon}}\Big)\sigma(1-x)$$

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$$\boldsymbol{u}_{\varepsilon n} = \sum_{j=0}^{n} \varepsilon^j \boldsymbol{u}_j, \quad \theta_{\varepsilon n} = \sum_{j=0}^{n} \varepsilon^j \theta_{j,\varepsilon}.$$

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The error function at order $n$ is

$$w_{n,\varepsilon} = u_\varepsilon - (u_{\varepsilon n} + \theta_{\varepsilon n}).$$

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The error function at order $n$ is

$$w_{n,\varepsilon} = u_\varepsilon - (u_{\varepsilon n} + \theta_{\varepsilon n}).$$

We are going to write the equation satisfied by $w_{n,\varepsilon}$ in order to deduce an error estimate with respect to $\varepsilon$.

We write the equation satisfied by each term:

$$\begin{aligned} u_0 &= f \\ -u''_{j-1} + u_j &= 0 \end{aligned} \implies -\varepsilon u''_{\varepsilon n} + u_{\varepsilon n} = f - \varepsilon^{n+1} u_{n+1} = f - \varepsilon^{n+1} f^{(2(n+1))}$$

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$$-\varepsilon\theta''_{j,\varepsilon} + \theta_{j,\varepsilon} = R_{j,\varepsilon} \implies -\varepsilon\theta''_{\varepsilon n} + \theta_{\varepsilon n} = R_{\varepsilon n}$$

where

$$R_{\varepsilon n} = \sum_{j=0}^{n} \varepsilon^j R_{j,\varepsilon}$$

$$R_{j,\varepsilon} = -\varepsilon\theta_j^l(\xi_l)\sigma''(x) - 2\varepsilon\frac{1}{\sqrt{\varepsilon}}(\theta_j^l)'(\xi_l)\sigma'(x) - \varepsilon\theta_j^r(\xi_r)\sigma''(1-x) - 2\varepsilon\frac{1}{\sqrt{\varepsilon}}(\theta_j^r)'(\xi_r)\sigma'(1-x).$$

Each $R_{j,\varepsilon}$ is an exponentially small term (as seen previously for $j=0$), thus

$$\|R_{\varepsilon n}\|_{L^2(0,1)} \leq C\varepsilon^m.$$

The equation for $w_{n,\varepsilon}$ is then

$$\begin{cases} -\varepsilon w''_{n,\varepsilon} + w_{n,\varepsilon} = \varepsilon^{n+1} f^{(2n+2)} - R_{\varepsilon n}, & 0 < x < 1 \\ w_{n,\varepsilon} = 0, & \text{at } x = 0, 1. \end{cases} \tag{28}$$

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Multiplying by $w_{n,\varepsilon}$ and integrating by parts, we obtain

$$\begin{aligned} \varepsilon \| w'_{n,\varepsilon} \|^2_{L^2(0,1)} + \| w_{n,\varepsilon} \|^2_{L^2(0,1)} &\leq \varepsilon^{n+1} \| f^{(2n+2)} \|_{L^2(0,1)} \| w_{n,\varepsilon} \|_{L^2(0,1)} + \| R_{\varepsilon n} \|_{L^2(0,1)} \| w_{n,\varepsilon} \|_{L^2(0,1)} \\ &\leq C \varepsilon^{n+1} \| w_{n,\varepsilon} \|_{L^2(0,1)} \\ &\leq \frac{C^2}{2} \varepsilon^{2(n+1)} + \frac{1}{2} \| w_{n,\varepsilon} \|^2_{L^2(0,1)} \end{aligned}$$

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Multiplying by $w_{n,\varepsilon}$ and integrating by parts, we obtain

$$\begin{aligned} \varepsilon \|w'_{n,\varepsilon}\|^2_{L^2(0,1)} + \|w_{n,\varepsilon}\|^2_{L^2(0,1)} &\leq \varepsilon^{n+1} \|f^{(2n+2)}\|_{L^2(0,1)} \|w_{n,\varepsilon}\|_{L^2(0,1)} + \|R_{\varepsilon n}\|_{L^2(0,1)} \|w_{n,\varepsilon}\|_{L^2(0,1)} \\ &\leq C\varepsilon^{n+1} \|w_{n,\varepsilon}\|_{L^2(0,1)} \\ &\leq \frac{C^2}{2} \varepsilon^{2(n+1)} + \frac{1}{2} \|w_{n,\varepsilon}\|^2_{L^2(0,1)} \end{aligned}$$

hence

$$\varepsilon \|w'_{n,\varepsilon}\|^2_{L^2(0,1)} + \|w_{n,\varepsilon}\|^2_{L^2(0,1)} \leq C\varepsilon^{2(n+1)}.$$

It follows, since $w_{n,\varepsilon} \in H^1_0(0,1)$, that

$$\|w_{n,\varepsilon}\|_{L^2(0,1)} \leq C\varepsilon^{n+1}$$

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## Theorem

*Assume that* $f \in H^{2n+2}(0,1)$. *Let* $u_\varepsilon$ *be the solution of*

$$\begin{cases} -\varepsilon u_\varepsilon'' + u_\varepsilon = f, \quad 0 < x < 1, \\ u_\varepsilon(0) = u_\varepsilon(1) = 0. \end{cases} \tag{29}$$

*Then there exists a constant* $C > 0$ *independent of* $\varepsilon$ *such that*

$$\| u_\varepsilon - (u_{\varepsilon n} + \theta_{\varepsilon n}) \|_{L^2(0,1)} \leq C \varepsilon^{n+1},$$

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## Remark

We can show the same result for the approximation using approximate correctors, i.e. replacing $\theta_{\varepsilon n}$ by

$$\sum_{j=0}^{n} \varepsilon^j \left( \theta_j^l \left( \frac{x}{\sqrt{\varepsilon}} \right) + \theta_j^r \left( \frac{1-x}{\sqrt{\varepsilon}} \right) \right),$$

since the difference is an exponentially small term.

An approximation of the solution was given by

$$u_\varepsilon(x) \approx u_0(x) + \theta_\varepsilon(x) = u_0(x) + \theta\left(\frac{x}{\varepsilon}\right)\sigma(x) \tag{30}$$

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We generalize the asymptotic expansions in powers of $\varepsilon$:

$$u_\varepsilon(x) \approx \sum_{j=0}^{\infty} \varepsilon^j \left(u_j + \theta_{j,\varepsilon}\right) \approx \sum_{j=0}^{\infty} \varepsilon^j \left(u_j(x) + \theta_j\left(\frac{x}{\varepsilon}\right)\right) \tag{31}$$

where $\theta_j(\xi) \underset{\xi\to\infty}{\to} 0$ and $\frac{d^k}{d\xi^k}\theta_j(\xi) \underset{\xi\to\infty}{\to} 0$.

We substitute in the convection-diffusion equation and identify each power of $\varepsilon$:

$$-\varepsilon\Big(\sum_{j=0}^{\infty}\varepsilon^j\Big(u_j''(x)+\frac{1}{\varepsilon^2}(\theta_j)''\Big(\frac{x}{\varepsilon}\Big)\Big)\Big)-\sum_{j=0}^{\infty}\varepsilon^j\Big(u_j'(x)+\frac{1}{\varepsilon}\theta_j'\Big(\frac{x}{\varepsilon}\Big)\Big)=f$$

# Asymptotic expansions

## Convection-diffusion equation

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$$\begin{cases}-u_0'=f\\ u_0(1)=0\end{cases}\qquad\begin{cases}-(\theta_1)''-\theta_1'=0\\ \theta_1(0)=-u_1(0)\end{cases}$$

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$$\begin{cases}-u_j'=u_{j-1}''=(-1)^j f^{(j)}\\ u_j(1)=0\end{cases}\qquad\begin{cases}-(\theta_{j+1})''-\theta_{j+1}'=0\\ \theta_{j+1}(0)=-u_{j+1}(0)\end{cases}$$

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We obtain $\theta_j(\xi)=-u_j(0)e^{-\xi}$.

We localize the approximate corrector and define the corrector

$$\theta_{j,\varepsilon}(x) = \theta_j\left(\frac{x}{\varepsilon}\right)\sigma(x).$$

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$$\boldsymbol{u}_{\varepsilon n} = \sum_{j=0}^{n} \varepsilon^j \boldsymbol{u}_j, \quad \theta_{\varepsilon n} = \sum_{j=0}^{n} \varepsilon^j \theta_{j,\varepsilon}.$$

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and

$$u_{\varepsilon n} = \sum_{j=0}^{n} \varepsilon^j u_j, \quad \theta_{\varepsilon n} = \sum_{j=0}^{n} \varepsilon^j \theta_{j,\varepsilon}.$$

The error function at order $n$ is

$$w_{n,\varepsilon} = u_\varepsilon - (u_{\varepsilon n} + \theta_{\varepsilon n}).$$

We are going to write the equation satisfied by $w_{n,\varepsilon}$ in order to deduce an error estimate with respect to $\varepsilon$.

We write the equation satisfied by each term:

$$\begin{aligned} -u_0' &= f \\ -u_{j-1}'' - u_j' &= 0 \end{aligned} \implies -\varepsilon u_{\varepsilon n}'' - u_{\varepsilon n}' = f + \varepsilon^{n+1} u_{n+1}' = f - \varepsilon^{n+1} (-1)^{n+1} f^{(n+1)}$$

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$$R_{\varepsilon n} = \sum_{j=0}^{n} \varepsilon^j R_{j,\varepsilon}$$

$$R_{j,\varepsilon} = -\varepsilon \theta_j(\xi)\sigma''(x) - 2\varepsilon \frac{1}{\varepsilon} \theta_j'(\xi)\sigma'(x) - \theta_j(\xi)\sigma'(x).$$

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Each $R_{j,\varepsilon}$ is an exponentially small term (as seen previously for $j = 0$), thus

$$\|R_{\varepsilon n}\|_{L^2(0,1)} \leq C\varepsilon^m.$$

The equation for $w_{n,\varepsilon}$ is then

$$\begin{cases} -\varepsilon w''_{n,\varepsilon} - w'_{n,\varepsilon} = \varepsilon^{n+1}(-1)^{n+1} f^{(n+1)} - R_{\varepsilon n}, & 0 < x < 1 \\ w_{n,\varepsilon} = 0, & \text{at } x = 0, 1. \end{cases} \tag{32}$$

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Multiplying by $w_{n,\varepsilon} e^x$ and integrating by parts, we obtain

$$\begin{aligned} \varepsilon \| w'_{n,\varepsilon} \|^2_{L^2(0,1)} + \frac{1-\varepsilon}{2} \| w_{n,\varepsilon} \|^2_{L^2(0,1)} &\leq C\varepsilon^{n+1} \| f^{(n+1)} \|_{L^2(0,1)} \| w_{n,\varepsilon} \|_{L^2(0,1)} + C \| R_{\varepsilon n} \|_{L^2(0,1)} \| w_{n,\varepsilon} \|_{L} \\ &\leq C\varepsilon^{n+1} \| w_{n,\varepsilon} \|_{L^2(0,1)} \\ &\leq C^2 \varepsilon^{2(n+1)} + \frac{1}{4} \| w_{n,\varepsilon} \|^2_{L^2(0,1)} \end{aligned}$$

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*Assume that* $f \in H^{n+1}(0,1)$. *Let* $u_\varepsilon$ *be the solution of*

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# Overview

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# Errata

Some errors found their way into the course videos: some were corrected in the corresponding slide, others are given/corrected below. Viewers are encouraged to find them by themselves.

- $\left|\varepsilon \int_0^1 u_\varepsilon' v'\right| \leq \sqrt{\varepsilon}\|\sqrt{\varepsilon} u_\varepsilon'\|_{L^2}\|v'\|_{L^2} \leq C\sqrt{\varepsilon} \to 0$ when $\varepsilon \to 0$. (video related to slide 9)
- For $\theta^I(\frac{x}{\sqrt{\varepsilon}}) = -u_0(0)e^{-\frac{x}{\sqrt{\varepsilon}}}$ for instance, $(\theta^I)'(\frac{x}{\sqrt{\varepsilon}}) = u_0(0)e^{-\frac{x}{\sqrt{\varepsilon}}}$ since $\theta^I(\xi) = -u_0(0)e^{-\xi}$. (video related to slide 13)
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- In high dimension (last video), under suitable assumptions, $\nabla\overline{W_0}\cdot v + \alpha\overline{W_0} = g$ is well posed in $\omega$ if we impose the value of $\overline{W_0}$ on $\Gamma_- = \{x \in \partial\omega,\ v\cdot\nu_\omega \leq 0\}$ (boundary part such that $v$ is going inward) (or if we impose the value of $\overline{W_0}$ on $\Gamma_+ = \{x \in \partial\omega,\ v\cdot\nu_\omega \geq 0\}$ (boundary part such that $v$ is going outward)). We cannot, in general, impose the value of $\overline{W_0}$ everywhere on $\partial\omega$ (cf. 1d case).
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*Thank you for your attention*

*Feel free to send me any questions or comments*

# Introduction to singular perturbations and boundary layers

## Penalization method for Neumann or Robin boundary conditions

Bouchra Bensiali

August 6, 2024

## A Boundary layer approach in the one-dimensional case

The one-dimensional case was studied in [1]. Here, we present an alternative proof for the convergence of the penalization method based on a boundary layer approach adapted from the one used in [3] for Dirichlet boundary conditions. The advantage of the boundary layer approach is that it is generalizable in higher dimension [2], unlike the approach used in [1] which was based on the explicit computation of the solution.

We thus consider the following one-dimensional problem with Neumann or Robin boundary conditions at $x = 1$

$$\begin{cases} -u'' + u = f \quad \text{in } \mathcal{U} = ]1,2[ \\ -u'(1) + \alpha u(1) = g(1), \quad u(2) = 0, \end{cases} \tag{1}$$

where $f \in L^2(]1,2[)$, $g(1) \in \mathbb{R}$ and $\alpha \geq 0$ are given. The corresponding penalized problem reads

$$\begin{cases} -u_\varepsilon'' + u_\varepsilon + \dfrac{\chi}{\varepsilon}(-u_\varepsilon' + \alpha u_\varepsilon - g(1)) = (1-\chi) f \quad \text{in } ]0,2[ \\ u_\varepsilon(0) = 0, \quad u_\varepsilon(2) = 0, \end{cases} \tag{2}$$

where $\varepsilon > 0$ is a small parameter (the penalization parameter), and $\chi$ is the characteristic function of the obstacle $\omega = ]0,1[$. The penalized problem (2) is equivalent to the following system

$$\begin{cases} -w_\varepsilon'' + w_\varepsilon + \dfrac{1}{\varepsilon}(-w_\varepsilon' + \alpha w_\varepsilon - g(1)) = 0 \quad \text{in } \omega & \text{(3a)} \\ -v_\varepsilon'' + v_\varepsilon = f \quad \text{in } \mathcal{U} & \text{(3b)} \\ w_\varepsilon(1) = v_\varepsilon(1) & \text{(3c)} \\ w_\varepsilon'(1) = v_\varepsilon'(1) & \text{(3d)} \\ w_\varepsilon(0) = 0 & \text{(3e)} \\ v_\varepsilon(2) = 0 & \text{(3f)} \end{cases}$$

where we distinguished the solution $w_\varepsilon$ inside the obstacle $\omega = ]0,1[$ from the solution $v_\varepsilon$ in the fluid domain $\mathcal{U} = ]1,2[$.

Based on our understanding of the one-dimensional case, we consider the following ansatz for the solution in terms of asymptotic expansions:

$$\begin{cases} v_\varepsilon(x) = V^0(x) + \varepsilon V^1(x) + \varepsilon^2 V^2(x) + \dots, \\ w_\varepsilon(x) = W^0\Big(x, \dfrac{x}{\varepsilon}\Big) + \varepsilon W^1\Big(x, \dfrac{x}{\varepsilon}\Big) + \varepsilon^2 W^2\Big(x, \dfrac{x}{\varepsilon}\Big) + \dots \end{cases} \tag{4}$$

where the profile terms inside the obstacle have the form

$$\begin{cases} W^i(x,z) = \overline{W}^i(x) + \theta(x)\widetilde{W}^i(x,z) \\ \text{where } \forall i,k \geq 0,\ \partial_z^k \widetilde{W}^i \xrightarrow[z\to+\infty]{} 0, \quad \forall x \in \omega \\ \text{and } \theta \in C^\infty(\bar{\omega}) \text{ such that } \theta \equiv 1 \text{ in a neighborhood } \omega_0 \text{ of } 0 \text{ and } \operatorname{supp}(\theta) \subset [0,\delta], \end{cases} \tag{5}$$

where $0 < \delta < 1$. This ansatz reflects the presence of a localized boundary layer near the boundary $x = 0$ and no boundary layer at the interface $x = 1$ between the fluid and the obstacle.

## A.1 Determination of profiles

For a function $(x,z) \mapsto W(x,z)$, we denote the derivatives with respect to the space variable $x$ as $W'$, $W''$, ... and the derivatives with respect to the boundary layer variable $z$ as $W_z$, $W_{zz}$, .... Thus the function $x \mapsto w(x) = W\big(x, \frac{x}{\varepsilon}\big)$ satisfies

$$\begin{cases} w'(x) = W'\Big(x, \dfrac{x}{\varepsilon}\Big) + \dfrac{1}{\varepsilon} W_z\Big(x, \dfrac{x}{\varepsilon}\Big) \\ w''(x) = W''\Big(x, \dfrac{x}{\varepsilon}\Big) + \dfrac{1}{\varepsilon}(W')_z\Big(x, \dfrac{x}{\varepsilon}\Big) + \dfrac{1}{\varepsilon}(W_z)'\Big(x, \dfrac{x}{\varepsilon}\Big) + \dfrac{1}{\varepsilon^2} W_{zz}\Big(x, \dfrac{x}{\varepsilon}\Big) \\ \qquad\quad = W''\Big(x, \dfrac{x}{\varepsilon}\Big) + \dfrac{2}{\varepsilon}(W_z)'\Big(x, \dfrac{x}{\varepsilon}\Big) + \dfrac{1}{\varepsilon^2} W_{zz}\Big(x, \dfrac{x}{\varepsilon}\Big). \end{cases} \tag{6}$$

In the following, we introduce formally the expressions (4) in the system (3a)–(3f) and identify the terms corresponding to each power of $\varepsilon$. To simplify the formal calculations, we assume in the following $\theta(x) = 1$ everywhere in (5) and an exponential decrease of $\partial_z^k \widetilde{W}^i(x,z)$ with respect to the variable $z$, this will be made rigorous afterwards.

### A.1.1 Asymptotic expansion of Eq. (3a) inside the obstacle $\omega$

- Order $\varepsilon^{-2}$: identifying the terms corresponding to the power $-2$ of $\varepsilon$ leads to

$$-W^0_{zz} - W^0_z = 0.$$

Using the decomposition (5) one obtains

$$-\overline{W}^0_{zz} - \widetilde{W}^0_{zz} - \overline{W}^0_z - \widetilde{W}^0_z = 0,$$

which leads to, as $\overline{W}^0$ does not depend on $z$,

$$-\widetilde{W}^0_{zz} - \widetilde{W}^0_z = 0 \quad \text{in } \omega \times \mathbb{R}^+. \tag{7}$$

- Order $\varepsilon^{-1}$:

$$-W^1_{zz} - 2(W^0_z)' - (W^0)' + \alpha W^0 - g(1) - W^1_z = 0,$$

which, by using hypothesis (5) and taking the limit when $z \to +\infty$, leads to

$$-(\overline{W}^0)' + \alpha \overline{W}^0 - g(1) = 0 \quad \text{in } \omega, \tag{8}$$

and by difference, we obtain

$$-\widetilde{W}^1_{zz} - 2(\widetilde{W}^0_z)' - (\widetilde{W}^0)' + \alpha \widetilde{W}^0 - \widetilde{W}^1_z = 0 \quad \text{in } \omega \times \mathbb{R}^+. \tag{9}$$

- Order $\varepsilon^0$:

$$-W^2_{zz} - 2(W^1_z)' - (W^0)'' + W^0 - W^2_z - (W^1)' + \alpha W^1 = 0,$$

and again using the same reasoning as above, by using hypothesis (5) and $z \to +\infty$, we obtain

$$-(\overline{W}^0)'' + \overline{W}^0 + \alpha \overline{W}^1 - (\overline{W}^1)' = 0 \quad \text{in } \omega, \tag{10}$$

and by difference,

$$-\widetilde{W}^2_{zz} - 2(\widetilde{W}^1_z)' - (\widetilde{W}^0)'' + \widetilde{W}^0 - \widetilde{W}^2_z - (\widetilde{W}^1)' + \alpha \widetilde{W}^1 = 0 \quad \text{in } \omega \times \mathbb{R}^+. \tag{11}$$

- Order $\varepsilon$: from

$$-W^3_{zz} - 2(W^2_z)' - (W^1)'' + W^1 - W^3_z - (W^2)' + \alpha W^2 = 0,$$

we deduce once again as before,

$$-(\overline{W}^1)'' + \overline{W}^1 - (\overline{W}^2)' + \alpha \overline{W}^2 = 0 \quad \text{in } \omega, \tag{12}$$

and by difference,

$$-\widetilde{W}^3_{zz} - 2(\widetilde{W}^2_z)' - (\widetilde{W}^1)'' + \widetilde{W}^1 - \widetilde{W}^3_z - (\widetilde{W}^2)' + \alpha \widetilde{W}^2 = 0 \quad \text{in } \omega \times \mathbb{R}^+. \tag{13}$$

### A.1.2 Asymptotic expansion of Eq. (3b) inside the fluid domain $\mathcal{U}$

- Order $\varepsilon^0$:
$$-(V^0)'' + V^0 = f \quad \text{in } \mathcal{U}. \tag{14}$$
- Order $\varepsilon^1$:
$$-(V^1)'' + V^1 = 0 \quad \text{in } \mathcal{U}. \tag{15}$$
- Order $\varepsilon^2$:
$$-(V^2)'' + V^2 = 0 \quad \text{in } \mathcal{U}. \tag{16}$$

### A.1.3 Asymptotic expansion of Eq. (3c) and (3d) (transmission conditions at $x = 1$)

We obtain simply, recalling the exponential decrease of the boundary layer terms

$$V^0(1) = \overline{W}^0(1),\ V^1(1) = \overline{W}^1(1),\ V^2(1) = \overline{W}^2(1). \tag{17}$$

and

$$(V^0)'(1) = (\overline{W}^0)'(1),\ (V^1)'(1) = (\overline{W}^1)'(1),\ (V^2)'(1) = (\overline{W}^2)'(1). \tag{18}$$

### A.1.4 Asymptotic expansion of Eq. (3e) and (3f) (Dirichlet boundary conditions at $x = 0$ and $x = 2$)

$$\begin{cases} \widetilde{W}^0(0,0) + \overline{W}^0(0) = 0 \\ \widetilde{W}^1(0,0) + \overline{W}^1(0) = 0 \\ \widetilde{W}^2(0,0) + \overline{W}^2(0) = 0 \end{cases} \tag{19}$$

and

$$V^0(2) = V^1(2) = V^2(2) = 0. \tag{20}$$

## A.2 Resolution of the profile equations

Here, we look for solutions to the equations determined previously, that will form the terms of the asymptotic expansion (4).

### A.2.1 Determination of $V^0$ and $W^0$

From (14), (8), (17), (18) and (20), we obtain that $V^0$ is solution to

$$\begin{cases} -(V^0)'' + V^0 = f & \text{in } \mathcal{U} = ]1,2[ \\ -(V^0)'(1) + \alpha V^0(1) = -(\overline{W}^0)'(1) + \alpha \overline{W}^0(1) = g(1), & V^0(2) = 0, \end{cases} \tag{21}$$

i.e. $V^0 = u$ the solution of the initial problem (1), and that $\overline{W}^0$ is solution to

$$\begin{cases} -(\overline{W}^0)' + \alpha \overline{W}^0 = g(1) & \text{in } \omega = ]0,1[ \\ \overline{W}^0(1) = V^0(1). \end{cases} \tag{22}$$

From (7) and the fact that $\widetilde{W}^0$ tends to 0 as $z \to +\infty$ , we obtain

$$\widetilde{W}^0(x,z) = w^0(x) e^{-z} \quad \text{in } \omega \times \mathbb{R}^+,$$

where $w^0$ is the value of $\widetilde{W}^0$ at $z = 0$. Using (19), we have $\widetilde{W}^0(0,0) = -\overline{W}^0(0)$. We extend this boundary condition to all $\omega$ and choose $\widetilde{W}^0(x,0) = -\overline{W}^0(0)$. This implies $w^0(x) = -\overline{W}^0(0)$ and thus

$$\widetilde{W}^0(x,z) = -\overline{W}^0(0) e^{-z} \quad \text{in } \omega \times \mathbb{R}^+. \tag{23}$$

#### A.2.2 Determination of $V^1$ and $W^1$

Similarly, from (15), (10), (17), (18) and (20), we obtain that $V^1$ is solution to

$$\begin{cases} -(V^1)'' + V^1 = 0 & \text{in } \mathcal{U} = ]1,2[ \\ -(V^1)'(1) + \alpha V^1(1) = -(\overline{W}^1)'(1) + \alpha \overline{W}^1(1) = (\overline{W}^0)''(1) - \overline{W}^0(1) \\ V^1(2) = 0, \end{cases} \tag{24}$$

and that $\overline{W}^1$ is solution to

$$\begin{cases} -(\overline{W}^1)' + \alpha \overline{W}^1 = (\overline{W}^0)'' - \overline{W}^0 & \text{in } \omega = ]0,1[ \\ \overline{W}^1(1) = V^1(1). \end{cases} \tag{25}$$

From (9), we have

$$-\widetilde{W}^1_{zz} - \widetilde{W}^1_z = 2(\widetilde{W}^0_z)' + (\widetilde{W}^0)' - \alpha \widetilde{W}^0 = \alpha \overline{W}^0(0) e^{-z} \quad \text{in } \omega \times \mathbb{R}^+,$$

using the obtained expression of $\widetilde{W}^0$ (23). The solution which tends to 0 as $z \to +\infty$ is given by

$$\widetilde{W}^1(x,z) = w^1(x) e^{-z} + \alpha \overline{W}^0(0) z e^{-z} \quad \text{in } \omega \times \mathbb{R}^+,$$

where $w^1$ is the value of $\widetilde{W}^1$ at $z = 0$, to be determined. Again, using (19), we have $\widetilde{W}^1(0,0) = -\overline{W}^1(0)$. We extend this boundary condition to all $\omega$ by setting $\widetilde{W}^1(x,0) = -\overline{W}^1(0)$. This choice leads to $w^1(x) = -\overline{W}^1(0)$ and thus

$$\widetilde{W}^1(x,z) = -\overline{W}^1(0) e^{-z} + \alpha \overline{W}^0(0) z\, e^{-z} \quad \text{in } \omega \times \mathbb{R}^+. \tag{26}$$

#### A.2.3 Determination of $V^2$ and $W^2$

Using the same reasoning as before, from (16), (12), (17), (18) and (20), we obtain that $V^2$ is solution to

$$\begin{cases} -(V^2)'' + V^2 = 0 & \text{in } \mathcal{U} = ]1,2[ \\ -(V^2)'(1) + \alpha V^2(1) = -(\overline{W}^2)'(1) + \alpha \overline{W}^2(1) = (\overline{W}^1)''(1) - \overline{W}^1(1) \\ V^2(2) = 0, \end{cases} \tag{27}$$

and that $\overline{W}^2$ is solution to

$$\begin{cases} -(\overline{W}^2)' + \alpha \overline{W}^2 = (\overline{W}^1)'' - \overline{W}^1 & \text{in } \omega = ]0,1[ \\ \overline{W}^2(1) = V^2(1). \end{cases} \tag{28}$$

From (11), we have

$$\begin{aligned} -\widetilde{W}^2_{zz} - \widetilde{W}^2_z &= 2(\widetilde{W}^1_z)' + (\widetilde{W}^0)'' - \widetilde{W}^0 + (\widetilde{W}^1)' - \alpha \widetilde{W}^1 \quad \text{in } \omega \times \mathbb{R}^+ \\ &= -\widetilde{W}^0 - \alpha \widetilde{W}^1 \\ &= \overline{W}^0(0) e^{-z} + \alpha \overline{W}^1(0) e^{-z} - \alpha^2 \overline{W}^0(0) z\, e^{-z} \\ &= (\overline{W}^0(0) + \alpha \overline{W}^1(0)) e^{-z} - \alpha^2 \overline{W}^0(0) z\, e^{-z} \\ &= c\, e^{-z} + d\, z\, e^{-z} \end{aligned}$$

using the obtained expressions of $\widetilde{W}^0$ (23) and $\widetilde{W}^1$ (26). The solution which tends to 0 as $z \to +\infty$ is given by

$$\widetilde{W}^2(x,z) = w^2(x) e^{-z} + (c+d) z\, e^{-z} + \frac{d}{2} z^2\, e^{-z} \quad \text{in } \omega \times \mathbb{R}^+,$$

where $w^2$ is the value of $\widetilde{W}^2$ at $z = 0$, to be determined. Once again, using (19), we have $\widetilde{W}^2(0,0) = -\overline{W}^2(0)$. We extend this boundary condition to all $\omega$ by setting $\widetilde{W}^2(x,0) = -\overline{W}^2(0)$. This choice leads to $w^2(x) = -\overline{W}^2(0)$ and thus

$$\begin{aligned} \widetilde{W}^2(x,z) = -\overline{W}^2(0) e^{-z} &+ (\overline{W}^0(0) + \alpha \overline{W}^1(0) - \alpha^2 \overline{W}^0(0)) z\, e^{-z} \\ &- \frac{\alpha^2}{2} \overline{W}^0(0) z^2\, e^{-z} \quad \text{in } \omega \times \mathbb{R}^+. \end{aligned} \tag{29}$$

In this one-dimensional setting, all the profile equations are well-posed with suitable regularity: $V^0 \in C^1(\overline{\mathcal{U}})$, $V^1, V^2 \in C^\infty(\overline{\mathcal{U}})$, $\overline{W}^i \in C^\infty(\overline{\omega})$ and $\widetilde{W}^i \in C^\infty(\overline{\omega} \times \mathbb{R}^+)$.

## A.3 Convergence of the asymptotic expansion

We search for a solution of the penalized problem (2) in the following form

$$\begin{cases} w_\varepsilon(x) = \theta(x)\widetilde{W}^0\Big(x, \dfrac{x}{\varepsilon}\Big) + \varepsilon\theta(x)\widetilde{W}^1\Big(x, \dfrac{x}{\varepsilon}\Big) + \varepsilon^2\theta(x)\widetilde{W}^2\Big(x, \dfrac{x}{\varepsilon}\Big) \\ \qquad\qquad + \overline{W}^0(x) + \varepsilon\overline{W}^1(x) + \varepsilon^2\overline{W}^2(x) + \varepsilon w_\varepsilon^r(x) & \text{in } \omega \\ v_\varepsilon(x) = V^0(x) + \varepsilon V^1(x) + \varepsilon^2 V^2(x) + \varepsilon v_\varepsilon^r(x) & \text{in } \mathcal{U}, \end{cases} \tag{30}$$

were $\widetilde{W}^i$, $\overline{W}^i$ and $V^i$ are the profile terms constructed previously, and $w_\varepsilon^r$ and $v_\varepsilon^r$ are the remainder terms that we will estimate in the following.

We use the following notations:

$$\begin{cases} W_{\text{app}} = \theta\widetilde{w}^0 + \varepsilon\theta\widetilde{w}^1 + \varepsilon^2\theta\widetilde{w}^2 + \overline{W}^0 + \varepsilon\overline{W}^1 + \varepsilon^2\overline{W}^2 \\ V_{\text{app}} = V^0 + \varepsilon V^1 + \varepsilon^2 V^2, \end{cases} \tag{31}$$

where $\widetilde{w}^i(x) = \widetilde{W}^i\big(x, \frac{x}{\varepsilon}\big)$, so that

$$\begin{cases} w_\varepsilon = W_{\text{app}} + \varepsilon w_\varepsilon^r & \text{in } \omega \\ v_\varepsilon = V_{\text{app}} + \varepsilon v_\varepsilon^r & \text{in } \mathcal{U}. \end{cases}$$

The aim of the following subsections is to show that the remainders $w_\varepsilon^r$ and $v_\varepsilon^r$ are bounded in $H^1$ independently of $\varepsilon$, from which we will conclude that

$$\|v_\varepsilon - V^0\|_{H^1(\mathcal{U})} = O(\varepsilon), \tag{32}$$

that is the convergence of the solution of the penalized problem (2) towards the solution of the initial problem (1) inside the fluid domain. On the other hand, we will obtain that

$$\|w_\varepsilon - W_{\text{app}}\|_{H^1(\omega)} = O(\varepsilon), \tag{33}$$

that is the presence of a boundary layer near the left boundary (at $x = 0$). In particular far from the boundary we have

$$\|w_\varepsilon - \overline{W}^0\|_{H^1(]\delta,1[)} = O(\varepsilon), \tag{34}$$

and $\overline{W}^0$ is the limit solution in the complementary domain. We thus recover the results obtained in [1] using a boundary layer approach (in fact, we obtain a finer description of what happens in the obstacle domain, which was not carried out in [1]).

### A.3.1 Equations of the remainders

Using the equations satisfied by $w_\varepsilon$ and $v_\varepsilon$, the remainders satisfy the following system

$$\begin{cases} -w_\varepsilon^{r\prime\prime} + w_\varepsilon^r + \dfrac{1}{\varepsilon}(-w_\varepsilon^{r\prime} + \alpha w_\varepsilon^r) = R_{\text{obst}}^\varepsilon & \text{in } \omega & (35a) \\ -v_\varepsilon^{r\prime\prime} + v_\varepsilon^r = R_{\text{flu}}^\varepsilon & \text{in } \mathcal{U} & (35b) \\ w_\varepsilon^r(1) = v_\varepsilon^r(1) & & (35c) \\ w_\varepsilon^{r\prime}(1) = v_\varepsilon^{r\prime}(1) & & (35d) \\ w_\varepsilon^r(0) = R_{\text{boundary}}^\varepsilon & & (35e) \\ v_\varepsilon^r(2) = 0, & & (35f) \end{cases}$$

where

$$\begin{aligned} R_{\text{obst}}^\varepsilon &= \frac{1}{\varepsilon}W''_{\text{app}} - \frac{1}{\varepsilon}W_{\text{app}} + \frac{1}{\varepsilon^2}W'_{\text{app}} - \frac{\alpha}{\varepsilon^2}W_{\text{app}} + \frac{g(1)}{\varepsilon^2} \\ &= \frac{1}{\varepsilon}W''_{\text{app}} + \frac{1}{\varepsilon^2}W'_{\text{app}} - \Big(\frac{1}{\varepsilon} + \frac{\alpha}{\varepsilon^2}\Big)W_{\text{app}} + \frac{g(1)}{\varepsilon^2} \\ R_{\text{flu}}^\varepsilon &= \frac{1}{\varepsilon}(f + V''_{\text{app}} - V_{\text{app}}) \\ &= \frac{1}{\varepsilon}(f + V^{0\prime\prime} + \varepsilon V^{1\prime\prime} + \varepsilon^2 V^{2\prime\prime} - V^0 - \varepsilon V^1 - \varepsilon^2 V^2) = 0 \\ R_{\text{boundary}}^\varepsilon &= -\frac{1}{\varepsilon}W_{\text{app}}(0) \\ &= -\frac{1}{\varepsilon}(\widetilde{W}^0(0,0) + \varepsilon\widetilde{W}^1(0,0) + \varepsilon^2\widetilde{W}^2(0,0) + \overline{W}^0(0) + \varepsilon\overline{W}^1(0) + \varepsilon^2\overline{W}^2(0)) = 0. \end{aligned}$$

To estimate the remainders, we need to estimate the right-hand sides of the equations, namely $R^\varepsilon_{\text{obst}}$. Using (31), one has

$$\begin{aligned}
W'_{\text{app}} &= \theta'\widetilde{w}^0 + \theta\widetilde{w}^{0'} + \varepsilon\theta'\widetilde{w}^1 + \varepsilon\theta\widetilde{w}^{1'} + \varepsilon^2\theta'\widetilde{w}^2 + \varepsilon^2\theta\widetilde{w}^{2'} + \overline{W}^{0'} + \varepsilon\overline{W}^{1'} + \varepsilon^2\overline{W}^{2'}\\
W''_{\text{app}} &= \theta''\widetilde{w}^0 + 2\theta'\widetilde{w}^{0'} + \theta\widetilde{w}^{0''} + \varepsilon\theta''\widetilde{w}^1 + 2\varepsilon\theta'\widetilde{w}^{1'} + \varepsilon\theta\widetilde{w}^{1''} + \varepsilon^2\theta''\widetilde{w}^2 + 2\varepsilon^2\theta'\widetilde{w}^{2'} + \varepsilon^2\theta\widetilde{w}^{2''}\\
&\quad + \overline{W}^{0''} + \varepsilon\overline{W}^{1''} + \varepsilon^2\overline{W}^{2''}.
\end{aligned}$$

Using (6) and the fact that $\widetilde{W}^i$ depend only on $z$, we obtain

$$\begin{aligned}
R^\varepsilon_{\text{obst}} &= \frac{1}{\varepsilon}(\theta''\widetilde{W}^0 + 2\theta'\frac{\widetilde{W}^0_z}{\varepsilon} + \theta\frac{\widetilde{W}^0_{zz}}{\varepsilon^2} + \varepsilon\theta''\widetilde{W}^1 + 2\varepsilon\theta'\frac{\widetilde{W}^1_z}{\varepsilon} + \varepsilon\theta\frac{\widetilde{W}^1_{zz}}{\varepsilon^2} + \varepsilon^2\theta''\widetilde{W}^2 + 2\varepsilon^2\theta'\frac{\widetilde{W}^2_z}{\varepsilon} + \varepsilon^2\theta\frac{\widetilde{W}^2_{zz}}{\varepsilon^2}\\
&\quad + \overline{W}^{0''} + \varepsilon\overline{W}^{1''} + \varepsilon^2\overline{W}^{2''}) + \frac{1}{\varepsilon^2}(\theta'\widetilde{W}^0 + \theta\frac{\widetilde{W}^0_z}{\varepsilon} + \varepsilon\theta'\widetilde{W}^1 + \varepsilon\theta\frac{\widetilde{W}^1_z}{\varepsilon} + \varepsilon^2\theta'\widetilde{W}^2 + \varepsilon^2\theta\frac{\widetilde{W}^2_z}{\varepsilon} + \overline{W}^{0'}\\
&\quad + \varepsilon\overline{W}^{1'} + \varepsilon^2\overline{W}^{2'}) - \Big(\frac{1}{\varepsilon} + \frac{\alpha}{\varepsilon^2}\Big)(\theta\widetilde{W}^0 + \varepsilon\theta\widetilde{W}^1 + \varepsilon^2\theta\widetilde{W}^2 + \overline{W}^0 + \varepsilon\overline{W}^1 + \varepsilon^2\overline{W}^2) + \frac{g(1)}{\varepsilon^2}\\
&= \frac{\theta''}{\varepsilon}\widetilde{W}^0 + \frac{2\theta'}{\varepsilon^2}\widetilde{W}^0_z + \frac{\theta}{\varepsilon^3}\widetilde{W}^0_{zz} + \theta''\widetilde{W}^1 + \frac{2\theta'}{\varepsilon}\widetilde{W}^1_z + \frac{\theta}{\varepsilon^2}\widetilde{W}^1_{zz} + \varepsilon\theta''\widetilde{W}^2 + 2\theta'\widetilde{W}^2_z + \frac{\theta}{\varepsilon}\widetilde{W}^2_{zz}\\
&\quad + \frac{1}{\varepsilon}\overline{W}^{0''} + \overline{W}^{1''} + \varepsilon\overline{W}^{2''} + \frac{\theta'}{\varepsilon^2}\widetilde{W}^0 + \frac{\theta}{\varepsilon^3}\widetilde{W}^0_z + \frac{\theta'}{\varepsilon}\widetilde{W}^1 + \frac{\theta}{\varepsilon^2}\widetilde{W}^1_z + \theta'\widetilde{W}^2 + \frac{\theta}{\varepsilon}\widetilde{W}^2_z + \frac{1}{\varepsilon^2}\overline{W}^{0'}\\
&\quad + \frac{1}{\varepsilon}\overline{W}^{1'} + \overline{W}^{2'} - \frac{\theta}{\varepsilon}\widetilde{W}^0 - \theta\widetilde{W}^1 - \varepsilon\theta\widetilde{W}^2 - \frac{1}{\varepsilon}\overline{W}^0 - \overline{W}^1 - \varepsilon\overline{W}^2\\
&\quad - \frac{\alpha\theta}{\varepsilon^2}\widetilde{W}^0 - \frac{\alpha\theta}{\varepsilon}\widetilde{W}^1 - \alpha\theta\widetilde{W}^2 - \frac{\alpha}{\varepsilon^2}\overline{W}^0 - \frac{\alpha}{\varepsilon}\overline{W}^1 - \alpha\overline{W}^2 + \frac{g(1)}{\varepsilon^2}.
\end{aligned}$$

Using the equations satisfied by the profile terms, different terms simplify in the previous expression and it remains

$$\begin{aligned}
R^\varepsilon_{\text{obst}} &= \frac{\theta''}{\varepsilon}\widetilde{W}^0 + \frac{2\theta'}{\varepsilon^2}\widetilde{W}^0_z + \theta''\widetilde{W}^1 + \frac{2\theta'}{\varepsilon}\widetilde{W}^1_z + \varepsilon\theta''\widetilde{W}^2 + 2\theta'\widetilde{W}^2_z + \varepsilon\overline{W}^{2''}\\
&\quad + \frac{\theta'}{\varepsilon^2}\widetilde{W}^0 + \frac{\theta'}{\varepsilon}\widetilde{W}^1 + \theta'\widetilde{W}^2 - \theta\widetilde{W}^1 - \varepsilon\theta\widetilde{W}^2 - \varepsilon\overline{W}^2 - \alpha\theta\widetilde{W}^2.
\end{aligned}$$

Since $\widetilde{W}^i(x,z)$ are of the form $P_i(z)e^{-z}$ where $P_i$ is a polynomial, and $P_i(z)e^{-z}$ are bounded in $\mathbb{R}^+$, $\widetilde{W}^i\big(x,\frac{x}{\varepsilon}\big)$ are bounded in $\omega$ independently of $\varepsilon$, thus we have $\forall x \in \omega$

$$\Big|-\theta(x)\widetilde{W}^1\Big(x,\frac{x}{\varepsilon}\Big) - \alpha\theta(x)\widetilde{W}^2\Big(x,\frac{x}{\varepsilon}\Big)\Big| \le C,$$

and

$$\Big|-\varepsilon\theta(x)\widetilde{W}^2\Big(x,\frac{x}{\varepsilon}\Big)\Big| \le C\varepsilon.$$

Also,

$$\Big|\varepsilon\overline{W}^{2''}(x) - \varepsilon\overline{W}^2(x)\Big| \le C\varepsilon.$$

The remaining terms are of the form $\frac{1}{\varepsilon^j}\theta'(x)\widetilde{W}^i\big(x,\frac{x}{\varepsilon}\big)$ or $\frac{1}{\varepsilon^j}\theta'(x)\widetilde{W}^i_z\big(x,\frac{x}{\varepsilon}\big)$ or $\frac{1}{\varepsilon^j}\theta''(x)\widetilde{W}^i\big(x,\frac{x}{\varepsilon}\big)$ or $\frac{1}{\varepsilon^j}\theta''(x)\widetilde{W}^i_z\big(x,\frac{x}{\varepsilon}\big)$ for $j \in \mathbb{Z}$. $\widetilde{W}^i_z(x,z)$ are also of the form $Q_i(z)e^{-z}$ where $Q_i$ is a polynomial. The terms under study are thus constituted of terms of the from $C\frac{1}{\varepsilon^j}\theta'(x)\big(\frac{x}{\varepsilon}\big)^k e^{-\frac{x}{\varepsilon}}$ or $C\frac{1}{\varepsilon^j}\theta''(x)\big(\frac{x}{\varepsilon}\big)^k e^{-\frac{x}{\varepsilon}}$ for $k \in \mathbb{N}$. Since $\theta \equiv 1$ in a neighborhood of 0, these terms are zero if $x \le \delta_0$ for a well-chosen $\delta_0 > 0$. These terms are thus bounded by

$$C\frac{1}{\varepsilon^{j+k}}e^{-\frac{\delta_0}{\varepsilon}} \le C\varepsilon,$$

since for all $\ell \in \mathbb{Z}$, $\varepsilon^\ell e^{-\frac{\delta_0}{\varepsilon}} \to 0$ as $\varepsilon \to 0^+$.

In conclusion, we have the following estimate

$$\|R^\varepsilon_{\text{obst}}\|_{L^2(\omega)} \le C. \tag{36}$$

In the following, we will need a finer estimate: since $R^\varepsilon_{\text{obst}}$ reduces to $\varepsilon\overline{W}^{2''} - \varepsilon\overline{W}^2$ outside of the support of $\theta$, we have

$$\|R^\varepsilon_{\text{obst}}\|_{L^2(]0,\delta[)} \le C \tag{37}$$
$$\|R^\varepsilon_{\text{obst}}\|_{L^2(]\delta,1[)} \le C\varepsilon. \tag{38}$$

### A.3.2 Estimate of the remainders

The last step is to estimate the remainders using some energy estimates. Unfortunately, multiplying by the remainders and integrating by parts yields an interface term of the form $\frac{1}{\varepsilon}(w_\varepsilon^r)^2(1)$ (with the wrong sign) that is not easy to control. Inspired by the approaches used in boundary layer methods in the case of advection-diffusion problems [6, 5, 4] where we multiply by test functions of the form $w_\varepsilon^r(x)e^{\pm x}$, we will rather multiply by weighted remainders as test functions where the weights have to be determined in order to get rid of the interface terms (between the fluid and the obstacle). More precisely, we multiply Eq. (35b) by $v_\varepsilon^r p_\varepsilon$ and we integrate over $\mathcal{U} = ]1,2[$, this yields

$$\begin{aligned}
-\int_1^2 v_\varepsilon^{r\prime\prime} v_\varepsilon^r p_\varepsilon + \int_1^2 v_\varepsilon^{r2} p_\varepsilon &= 0\\
\int_1^2 v_\varepsilon^{r\prime}(v_\varepsilon^r p_\varepsilon)' - [v_\varepsilon^{r\prime} v_\varepsilon^r p_\varepsilon]_1^2 + \int_1^2 v_\varepsilon^{r2} p_\varepsilon &= 0\\
\int_1^2 v_\varepsilon^{r\prime} v_\varepsilon^{r\prime} p_\varepsilon + \int_1^2 v_\varepsilon^{r\prime} v_\varepsilon^r p_\varepsilon' + v_\varepsilon^{r\prime}(1) v_\varepsilon^r(1) p_\varepsilon(1) + \int_1^2 v_\varepsilon^{r2} p_\varepsilon &= 0\\
\int_1^2 v_\varepsilon^{r\prime 2} p_\varepsilon + \int_1^2 \Big(\frac{v_\varepsilon^{r2}}{2}\Big)' p_\varepsilon' + v_\varepsilon^{r\prime}(1) v_\varepsilon^r(1) p_\varepsilon(1) + \int_1^2 v_\varepsilon^{r2} p_\varepsilon &= 0\\
\int_1^2 v_\varepsilon^{r\prime 2} p_\varepsilon - \int_1^2 \frac{v_\varepsilon^{r2}}{2} p_\varepsilon'' + \Big[\frac{v_\varepsilon^{r2}}{2} p_\varepsilon'\Big]_1^2 + v_\varepsilon^{r\prime}(1) v_\varepsilon^r(1) p_\varepsilon(1) + \int_1^2 v_\varepsilon^{r2} p_\varepsilon &= 0\\
\int_1^2 v_\varepsilon^{r\prime 2} p_\varepsilon - \int_1^2 \frac{v_\varepsilon^{r2}}{2} p_\varepsilon'' - \frac{v_\varepsilon^{r2}(1)}{2} p_\varepsilon'(1) + v_\varepsilon^{r\prime}(1) v_\varepsilon^r(1) p_\varepsilon(1) + \int_1^2 v_\varepsilon^{r2} p_\varepsilon &= 0\\
\int_1^2 v_\varepsilon^{r\prime 2} p_\varepsilon + \int_1^2 \Big(p_\varepsilon - \frac{p_\varepsilon''}{2}\Big) v_\varepsilon^{r2} - \frac{v_\varepsilon^{r2}(1)}{2} p_\varepsilon'(1) + v_\varepsilon^{r\prime}(1) v_\varepsilon^r(1) p_\varepsilon(1) &= 0. \qquad (E_1)
\end{aligned}$$

Similarly, we multiply Eq. (35a) by $w_\varepsilon^r q_\varepsilon$ and we integrate over $\omega = ]0,1[$, we obtain

$$\begin{aligned}
-\int_0^1 w_\varepsilon^{r\prime\prime} w_\varepsilon^r q_\varepsilon + \int_0^1 w_\varepsilon^{r2} q_\varepsilon - \frac{1}{\varepsilon}\int_0^1 w_\varepsilon^{r\prime} w_\varepsilon^r q_\varepsilon + \frac{\alpha}{\varepsilon}\int_0^1 w_\varepsilon^{r2} q_\varepsilon &= \int_0^1 R_{\mathrm{obst}}^\varepsilon w_\varepsilon^r q_\varepsilon\\
\int_0^1 w_\varepsilon^{r\prime}(w_\varepsilon^r q_\varepsilon)' - [w_\varepsilon^{r\prime} w_\varepsilon^r q_\varepsilon]_0^1 + \int_0^1 w_\varepsilon^{r2} q_\varepsilon - \frac{1}{\varepsilon}\int_0^1 w_\varepsilon^{r\prime} w_\varepsilon^r q_\varepsilon + \frac{\alpha}{\varepsilon}\int_0^1 w_\varepsilon^{r2} q_\varepsilon &= \int_0^1 R_{\mathrm{obst}}^\varepsilon w_\varepsilon^r q_\varepsilon\\
\int_0^1 w_\varepsilon^{r\prime 2} q_\varepsilon + \int_0^1 w_\varepsilon^{r\prime} w_\varepsilon^r q_\varepsilon' - w_\varepsilon^{r\prime}(1) w_\varepsilon^r(1) q_\varepsilon(1) + \int_0^1 w_\varepsilon^{r2} q_\varepsilon - \frac{1}{\varepsilon}\int_0^1 w_\varepsilon^{r\prime} w_\varepsilon^r q_\varepsilon + \frac{\alpha}{\varepsilon}\int_0^1 w_\varepsilon^{r2} q_\varepsilon &= \int_0^1 R_{\mathrm{obst}}^\varepsilon w_\varepsilon^r q_\varepsilon\\
\int_0^1 w_\varepsilon^{r\prime 2} q_\varepsilon + \int_0^1 \Big(q_\varepsilon' - \frac{q_\varepsilon}{\varepsilon}\Big) w_\varepsilon^{r\prime} w_\varepsilon^r + \Big(1 + \frac{\alpha}{\varepsilon}\Big)\int_0^1 w_\varepsilon^{r2} q_\varepsilon - w_\varepsilon^{r\prime}(1) w_\varepsilon^r(1) q_\varepsilon(1) &= \int_0^1 R_{\mathrm{obst}}^\varepsilon w_\varepsilon^r q_\varepsilon\\
\int_0^1 w_\varepsilon^{r\prime 2} q_\varepsilon - \int_0^1 \frac{w_\varepsilon^{r2}}{2} + \Big[\frac{w_\varepsilon^{r2}}{2}\Big(q_\varepsilon'' - \frac{q_\varepsilon'}{\varepsilon}\Big)\Big]_0^1 + \Big(1 + \frac{\alpha}{\varepsilon}\Big)\int_0^1 w_\varepsilon^{r2} q_\varepsilon - w_\varepsilon^{r\prime}(1) w_\varepsilon^r(1) q_\varepsilon(1) &= \int_0^1 R_{\mathrm{obst}}^\varepsilon w_\varepsilon^r q_\varepsilon\\
\int_0^1 w_\varepsilon^{r\prime 2} q_\varepsilon + \int_0^1 w_\varepsilon^{r2}\Big(\Big(1 + \frac{\alpha}{\varepsilon}\Big) q_\varepsilon - \frac{q_\varepsilon''}{2} + \frac{q_\varepsilon'}{2\varepsilon}\Big) + \frac{w_\varepsilon^{r2}(1)}{2}\Big(q_\varepsilon'(1) - \frac{q_\varepsilon(1)}{\varepsilon}\Big) - w_\varepsilon^{r\prime}(1) w_\varepsilon^r(1) q_\varepsilon(1) &= \int_0^1 R_{\mathrm{obst}}^\varepsilon w_\varepsilon^r q_\varepsilon. \qquad (E_2)
\end{aligned}$$

By adding $(E_1)$ and$(E_2)$, and by choosing $p_\varepsilon$ and $q_\varepsilon$ such that the interface terms vanish, i.e.

$$\begin{cases} p_\varepsilon(1) = q_\varepsilon(1)\\ p_\varepsilon'(1) = q_\varepsilon'(1) - \dfrac{q_\varepsilon(1)}{\varepsilon} \end{cases} \qquad (39)$$

we obtain

$$\int_1^2 v_\varepsilon^{r\prime 2} p_\varepsilon + \int_1^2 \Big(p_\varepsilon - \frac{p_\varepsilon''}{2}\Big) v_\varepsilon^{r2} + \int_0^1 w_\varepsilon^{r\prime 2} q_\varepsilon + \int_0^1 w_\varepsilon^{r2}\Big(\Big(1 + \frac{\alpha}{\varepsilon}\Big) q_\varepsilon - \frac{q_\varepsilon''}{2} + \frac{q_\varepsilon'}{2\varepsilon}\Big) = \int_0^1 R_{\mathrm{obst}}^\varepsilon w_\varepsilon^r q_\varepsilon. \qquad (40)$$

**Assume there exists** $p_\varepsilon$ **and** $q_\varepsilon$ such that for sufficiently small $\varepsilon > 0$

$$\begin{cases} (39) \text{ is satisfied} & \\ p_\varepsilon \geq \beta > 0 & \text{in } \mathcal{U} = (1,2) \\ q_\varepsilon \geq \beta > 0 & \text{in } \omega = (0,1) \\ p_\varepsilon - \dfrac{p_\varepsilon''}{2} \geq 0 & \text{in } \mathcal{U} \\ \Big(1 + \dfrac{\alpha}{\varepsilon}\Big) q_\varepsilon - \dfrac{q_\varepsilon''}{2} + \dfrac{q_\varepsilon'}{2\varepsilon} \geq 0 & \text{in } \omega \\ \|q_\varepsilon\|_{L^\infty(0,\delta)} \leq C \text{ and } \|q_\varepsilon\|_{L^\infty(\delta,1)} \leq \dfrac{C}{\varepsilon}, & \end{cases} \tag{41}$$

then from (40), we deduce, using also (37) and (38), that

$$\begin{aligned} \beta\Big(\int_1^2 v_\varepsilon^{r\prime 2} + \int_0^1 w_\varepsilon^{r\prime 2}\Big) &\leq \|R_{\text{obst}}^\varepsilon q_\varepsilon\|_{L^2(0,1)} \|w_\varepsilon^r\|_{L^2(0,1)} \\ &\leq \Big(\|R_{\text{obst}}^\varepsilon q_\varepsilon\|_{L^2(0,\delta)} + \|R_{\text{obst}}^\varepsilon q_\varepsilon\|_{L^2(\delta,1)}\Big)\|w_\varepsilon^r\|_{L^2(0,1)} \\ &\leq \Big(\|R_{\text{obst}}^\varepsilon\|_{L^2(0,\delta)}\|q_\varepsilon\|_{L^\infty(0,\delta)} + \|R_{\text{obst}}^\varepsilon\|_{L^2(\delta,1)}\|q_\varepsilon\|_{L^\infty(\delta,1)}\Big)\|w_\varepsilon^r\|_{L^2(0,1)} \\ &\leq C\|w_\varepsilon^{r\prime}\|_{L^2(0,1)} \qquad \text{(by Poincaré inequality } (w_\varepsilon^r(0) = 0)) \\ \|v_\varepsilon^{r\prime}\|^2_{L^2(1,2)} + \|w_\varepsilon^{r\prime}\|^2_{L^2(0,1)} &\leq C\|w_\varepsilon^{r\prime}\|_{L^2(0,1)} \\ &\leq \frac{C^2}{2} + \frac{1}{2}\|w_\varepsilon^{r\prime}\|^2_{L^2(0,1)} \qquad \text{(by Young inequality)} \\ \|v_\varepsilon^{r\prime}\|^2_{L^2(1,2)} + \frac{1}{2}\|w_\varepsilon^{r\prime}\|^2_{L^2(0,1)} &\leq C. \end{aligned}$$

In conclusion, using again Poincaré inequality ($v_\varepsilon^r(2) = 0$), we have established that $\|v_\varepsilon^r\|_{H^1(1,2)} \leq C$ and $\|w_\varepsilon^r\|_{H^1(0,1)} \leq C$, and the convergence of the asymptotic expansion follows as explained in Subsection A.3.

**A.3.2.1 Construction of suitable weight functions** $p_\varepsilon$ **and** $q_\varepsilon$ We will look for supersolutions $p_\varepsilon$, $q_\varepsilon$ satisfying

$$\begin{cases} -q_\varepsilon'' + \dfrac{q_\varepsilon'}{\varepsilon} + q_\varepsilon = b_\varepsilon \geq 0 & \text{in } \omega = (0,1) \\ -p_\varepsilon'' + p_\varepsilon = a_\varepsilon \geq 0 & \text{in } \mathcal{U} = (1,2) \\ p_\varepsilon(1) = q_\varepsilon(1) & \\ p_\varepsilon'(1) = q_\varepsilon'(1) - \dfrac{q_\varepsilon(1)}{\varepsilon} & \\ p_\varepsilon \geq \beta > 0 & \text{in } \mathcal{U} = (1,2) \\ q_\varepsilon \geq \beta > 0 & \text{in } \omega = (0,1) \\ \|q_\varepsilon\|_{L^\infty(0,\delta)} \leq C \text{ and } \|q_\varepsilon\|_{L^\infty(\delta,1)} \leq \dfrac{C}{\varepsilon}. & \end{cases} \tag{42}$$

It is easy to see that the solutions to (42) satisfy (41) and thus yield suitable functions to show the convergence as explained before. First, (39) is satisfied, along with the positivity of $p_\varepsilon$, $q_\varepsilon \geq \beta > 0$ and the estimates on $\|q_\varepsilon\|_{L^\infty}$, thus we have

$$p_\varepsilon - \frac{p_\varepsilon''}{2} = \frac{p_\varepsilon}{2} + \frac{-p_\varepsilon'' + p_\varepsilon}{2} \geq \frac{p_\varepsilon}{2} \geq 0$$

$$\Big(1 + \frac{\alpha}{\varepsilon}\Big) q_\varepsilon - \frac{q_\varepsilon''}{2} + \frac{q_\varepsilon'}{2\varepsilon} = \frac{\alpha}{\varepsilon} q_\varepsilon + \frac{q_\varepsilon}{2} + \frac{1}{2}\Big(-q_\varepsilon'' + \frac{q_\varepsilon'}{\varepsilon} + q_\varepsilon\Big) \geq \Big(\frac{\alpha}{\varepsilon} + \frac{1}{2}\Big) q_\varepsilon \geq 0.$$

Let us first outline the link between (42) and a **dual problem** of the penalized problem. Let $\varphi \in H_0^1(\Omega)$, where $\Omega = (0,1)$, then by multiplying the equations (42) by $\varphi$ and integrating by parts, we obtain:

$$\int_0^1 q_\varepsilon' \varphi' - {q_\varepsilon}'(1)\varphi(1) - \frac{1}{\varepsilon}\int_0^1 q_\varepsilon \varphi' + \frac{1}{\varepsilon} q_\varepsilon(1)\varphi(1) + \int_0^1 q_\varepsilon \varphi = \int_0^1 b_\varepsilon \varphi \tag{43}$$

and

$$\int_1^2 p_\varepsilon' \varphi' + {p_\varepsilon}'(1)\varphi(1) + \int_1^2 p_\varepsilon \varphi = \int_1^2 a_\varepsilon \varphi. \tag{44}$$

Summing the two previous equations and using the transmission conditions, we obtain

$$\int_0^1 q_\varepsilon'\varphi' - \frac{1}{\varepsilon}\int_0^1 q_\varepsilon\varphi' + \int_0^1 q_\varepsilon\varphi + \int_1^2 p_\varepsilon'\varphi' + \int_1^2 p_\varepsilon\varphi = \int_0^1 b_\varepsilon\varphi + \int_1^2 a_\varepsilon\varphi. \tag{45}$$

If we denote by $r_\varepsilon$ the function whose restriction to $(0,1)$ is $q_\varepsilon$ and whose restriction to $(1,2)$ is $p_\varepsilon$, this means that $r_\varepsilon$ is solution to following variational formulation

$$\begin{cases}\text{Find } r_\varepsilon \in V = \{v \in H^1(0,1), v(0) \text{ and } v(2) \text{ are imposed}\} \text{ such that } \forall\varphi \in H_0^1(0,1)\\ \displaystyle\int_0^2 r_\varepsilon'\varphi' + \int_0^2 r_\varepsilon\varphi - \frac{1}{\varepsilon}\int_0^2 \chi r_\varepsilon\varphi' = \int_0^1 b_\varepsilon\varphi + \int_1^2 a_\varepsilon\varphi,\end{cases} \tag{46}$$

which is of the form of a dual problem to the penalized problem (with $\alpha = 0$) [2] where the advection term is in a conservative form. The existence and uniqueness of the solution to the dual problem was also established in [2] (it was used in the proof of the existence and uniqueness of the penalized problem).

*Remark* 1. Figure. 1 shows the solution of (46) if we impose for instance $a_\varepsilon(x) = 0$, $b_\varepsilon(x) = 0$, and $q_\varepsilon(0) = p_\varepsilon(2) = 1$. We can show (using explicit caluclations and asymptotic expansions with respect to $\varepsilon$ as in [1]) that the other conditions in (42) (positivity of $p_\varepsilon$, $q_\varepsilon$ and estimates on $q_\varepsilon$) are satisfied for $\varepsilon$ small enough, but we would like a construction not relying on explicit calculations, that could be extendable in higher dimension.

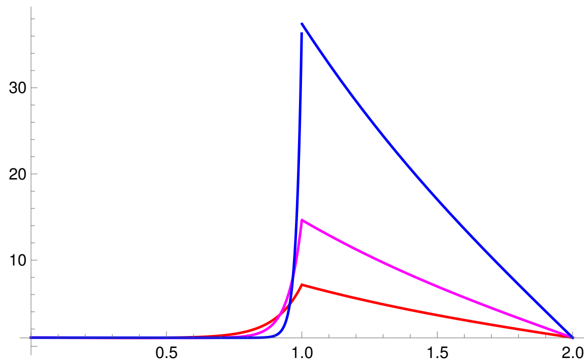


Figure 1: Plot of $q_\varepsilon$ in $(0,1)$ and $p_\varepsilon$ in $(1,2)$ solutions to the dual problem (46) with $a_\varepsilon = 0$, $b_\varepsilon = 0$ and $q_\varepsilon(0) = p_\varepsilon(2) = 1$, for different values of $\varepsilon$.

We now go back to exhibit suitable supersolutions satisfying (42): we take

$$q_\varepsilon(x) = 1 + \frac{1}{\varepsilon}e^{\frac{x-1}{\varepsilon}} - \frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon}} \quad \text{in } \omega = (0,1), \tag{47}$$

so that

$$q_\varepsilon(x) \geq 1 - \frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon}} \to 1 \text{ when } \varepsilon \to 0 \tag{48}$$

and

$$-q_\varepsilon'' + \frac{q_\varepsilon'}{\varepsilon} + q_\varepsilon = q_\varepsilon \geq 0. \tag{49}$$

To satisfy the transmission conditions, we choose an affine $p_\varepsilon$

$$p_\varepsilon(x) = q_\varepsilon(1) + \Big(q_\varepsilon'(1) - \frac{q_\varepsilon(1)}{\varepsilon}\Big)(x-1) \quad \text{in } \mathcal{U} = (1,2), \tag{50}$$

so that

$$p_\varepsilon(2) = q_\varepsilon(1) + q_\varepsilon'(1) - \frac{q_\varepsilon(1)}{\varepsilon} = \Big(1 - \frac{1}{\varepsilon}\Big)\Big(1 + \frac{1}{\varepsilon} - \frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon}}\Big) + \frac{1}{\varepsilon^2} = 1 + \frac{1}{\varepsilon^2}e^{-\frac{1}{\varepsilon}} \to 1 \text{ when } \varepsilon \to 0 \tag{51}$$

and

$$-p_\varepsilon'' + p_\varepsilon = p_\varepsilon \geq \min(p_\varepsilon(1), p_\varepsilon(2)) = \min(q_\varepsilon(1), p_\varepsilon(2)) \geq 0 \quad \text{for } \varepsilon \text{ small enough.} \tag{52}$$

Thus the strict positivity of $p_\varepsilon$ and $q_\varepsilon$ for sufficiently small $\varepsilon > 0$. Finally, for all $x \in (0,\delta)$ with $\delta < 1$,

$$|q_\varepsilon(x)| \leq 1 + \frac{1}{\varepsilon}e^{\frac{\delta-1}{\varepsilon}} - \frac{1}{\varepsilon}e^{-\frac{1}{\varepsilon}} \leq C$$

for sufficiently small $\varepsilon > 0$, and, for $x \in (\delta, 1)$,

$$|q_\varepsilon(x)| \leq 1 + \frac{1}{\varepsilon} - \frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon}} \leq \frac{C}{\varepsilon}$$

for sufficiently small $\varepsilon$, then the desired estimates on $\|q_\varepsilon\|_{L^\infty}$. This completes the existence of suitable weight functions $p_\varepsilon$, $q_\varepsilon$ in the one-dimensional case.

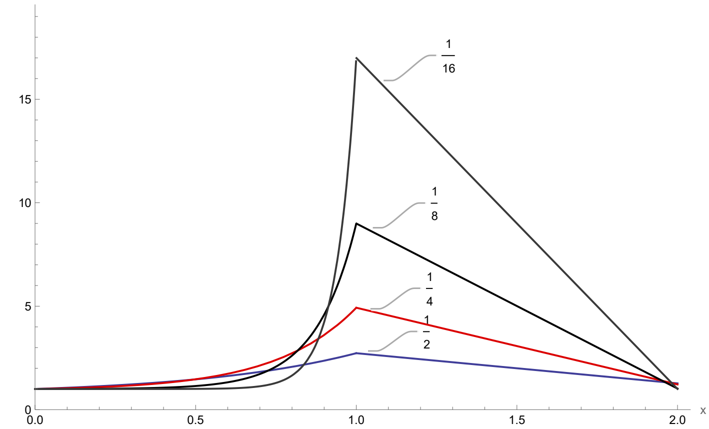


Figure 2: Plot of suitable supersolutions $q_\varepsilon$ (47) in $(0, 1)$ and $p_\varepsilon$ (50) in $(1, 2)$ satisfying (42) in the one-dimensional case, for different values of $\varepsilon$.

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