Residuated Lattices and Residuated Multilattices with Applications

Pr. Célestin Lélé, CIMPA-ICTP course

June 17, 2024

CIMPA

Course content

Chapter 1 : Lattices

Table of contents

Order relation

Let P be a set. An **order** or **partial order** on P is a binary relation "$\leq$" defined on P such that for all $a, b, c \in P$:

(i) $a \leq a$ (reflexivity);

(ii) $a \leq b$ and $b \leq a$ imply $a = b$ (antisymmetry);

(iii) $a \leq b$ and $b \leq c$ imply $a \leq c$ (transitivity).

Order relation

Let P be a set. An **order** or **partial order** on P is a binary relation "$\leq$" defined on P such that for all $a, b, c \in P$:

(i) $a \leq a$ (reflexivity);

(ii) $a \leq b$ and $b \leq a$ imply $a = b$ (antisymmetry);

(iii) $a \leq b$ and $b \leq c$ imply $a \leq c$ (transitivity).

- $(P, \leq)$ is called a **partially ordered set** or briefly a **poset**.

Order relation

Let P be a set. An **order** or **partial order** on P is a binary relation "$\leq$" defined on P such that for all $a, b, c \in P$:

(i) $a \leq a$ (reflexivity);

(ii) $a \leq b$ and $b \leq a$ imply $a = b$ (antisymmetry);

(iii) $a \leq b$ and $b \leq c$ imply $a \leq c$ (transitivity).

- $(P, \leq)$ is called a **partially ordered set** or briefly a **poset**.
- If the elements of P are pairwise comparable, i.e., $x \leq y$ or $y \leq x$, for all $x, y \in P$, then the order $\leq$ is **total** and $(P, \leq)$ is a **totally ordered set**, also called a **chain**.

Order relation

Let P be a set. An **order** or **partial order** on P is a binary relation "$\leq$" defined on P such that for all $a, b, c \in P$:

(i) $a \leq a$ (reflexivity);

(ii) $a \leq b$ and $b \leq a$ imply $a = b$ (antisymmetry);

(iii) $a \leq b$ and $b \leq c$ imply $a \leq c$ (transitivity).

- $(P, \leq)$ is called a **partially ordered set** or briefly a **poset**.
- If the elements of P are pairwise comparable, i.e., $x \leq y$ or $y \leq x$, for all $x, y \in P$, then the order $\leq$ is **total** and $(P, \leq)$ is a **totally ordered set**, also called a **chain**.
- From any partial order $\leq$ derives the relation "$<$" called **strict order**, and define by $x < y$ iff ($x \leq y$ and $x \neq y$).

Upper bound and lower bound

Let X be a subset of a Poset P. [1]

- An element $p \in P$ is called an **upper bound** of X if $a \leq p$ for every $a \in X$.
  We say that $p \in P$ is the least upper bound of P (l.u.b of P), or **supremum** of P ($supP$) if p is the smallest among the upper bounds of P.

[1]We will usually denote a poset without specifying the partially ordered set.

Upper bound and lower bound

Let X be a subset of a Poset P. [1]

- An element $p \in P$ is called an **upper bound** of X if $a \leq p$ for every $a \in X$.
  We say that $p \in P$ is the least upper bound of P (l.u.b of P), or **supremum** of P ($supP$) if p is the smallest among the upper bounds of P.
- A dual statement defines the greatest lower bound of P (g.l.b of P), also called the **infimum** of P ($infP$).

[1] We will usually denote a poset without specifying the partially ordered set.

Hasse diagram of a poset

An advantage of a poset P is that we can represent P by a configuration using dots for the elements of P, and having $x < y$ if there are upward line segments from x to y; this diagram is called **Hasse diagram**.

Hasse diagram of a poset

An advantage of a poset P is that we can represent P by a configuration using dots for the elements of P, and having $x < y$ if there are upward line segments from x to y; this diagram is called **Hasse diagram**.

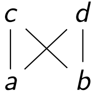


Figure 1: Hasse diagram of the poset $(P, \leq)$ with $P = \{a, b, c, d\}$ and $a, b < c, d$

Hasse diagram of a poset (cont.)

Exercise 1

1. *Prove that the relation "a divides b" if there exists an integer c such that* $ac = b$ *and is denoted by* a/b, *is a partial order relation on the set of all positive integers* $\mathbb{N}$.
2. *Let* P *be the set* $\{1, 3, 4, 12, 24, 48, 72\}$. *Show that* $(P, /)$ *is a poset and draw its Hasse diagram.*

Table of contents

Downset, Upset

Definition 2

A subset Q of a poset P is called **downset** *or order ideal if for all $x \in Q$ and $y \in P$, $y \leq x$ implies $y \in Q$. We write* $\downarrow Q = \{y \in P : y \leq x, \text{ for all } x \in Q\}$

Definition 3

A subset Q of a poset P is called an **upset** *or order filter if for all $x \in Q$ and $y \in P$, $x \leq y$ implies $y \in Q$.* $\uparrow Q = \{y \in P : x \leq y, \text{ for all } x \in Q\}$

Table of contents

What is a lattice?

Definition 4

A **lattice** *is a poset* $(L, \leq)$ *in which for every* $x, y \in L$*, both sup*$\{x, y\}$ *and inf*$\{x, y\}$ *exists in* L*.*

- This definition is known as lattice ordered set. There is an other way of defining a lattice : lattice as an algebraic structure.

What is a lattice (cont.)

Definition 5

A nonempty set P endowed with two binary operations $\vee$ and $\wedge$ ("join" and "meet" respectively) is called a **lattice** *if for every $x, y, z \in P$, the following conditions hold.*

$(L1)$: Commutative laws

(a) $x \vee y = y \vee x$

(b) $x \wedge y = y \wedge x$

$(L2)$: Associative laws

(a) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$

(b) $x \vee (y \vee z) = (x \vee y) \vee z$

$(L3)$: Idempotent laws

(a) $x \wedge x = x$

(b) $x \vee x = x$

$(L4)$: Absorption laws

(a) $x = x \wedge (x \vee y)$

(b) $x = x \vee (x \wedge y)$

- It should be noticed that these definitions are equivalent.

Hasse diagram of a lattice

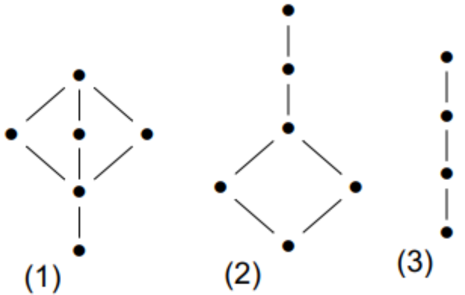


Figure 2: Examples of Hasse diagrams illustrating lattices within posets

Hasse diagram of posets that are not lattices

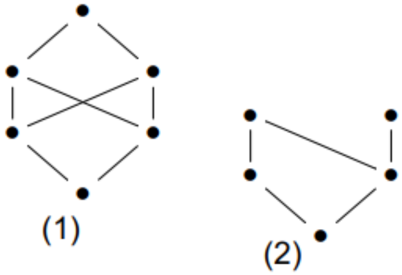


Figure 3: Examples of Hasse diagram of some posets that are not lattices

Some examples of lattices

Example 6

The power set $\mathcal{P}(X)$ of a set X is a lattice.

Some examples of lattices

Example 6

Let $\mathcal{C}[0,1]$ be the set of continuous functions from $[0,1]$ to the set of reals. We define "$\leq$" on $C[0,1]$ by $f \leq g$ iff $f(x) \leq g(x), \forall x \in [0,1]$. Then, "$\leq$" is a partial order which makes $C[0,1]$ to be a lattice.

Some examples of lattices

Example 6

If G is a group, and $N(G)$ the set of normal subgroups of G, then $(N(G), \wedge, \vee)$ is a lattice in which $\vee$ and $\wedge$ are defined by: $N_1 \wedge N_2 = N_1 \cap N_2$, $N_1 \vee N_2 = N_1 N_2 = \{n_1 n_2 : n_1 \in N_1, n_2 \in N_2\}$, for any $N_1, N_2 \in N(G)$.
Also, the set $S(G)$ of subgroups of G with the partial order $\subseteq$ forms a lattice.

Some examples of lattices

Example 6

Consider a ring R, and $\mathcal{I}(R)$ the set of ideals of R. Then, $(\mathcal{I}(R), \wedge, \vee)$ is a lattice with $I_1 \wedge I_2 = I_1 \cap I_2$, $I_1 \vee I_2 = I_1 + I_2 = \{i_1 + i_2 : i_1 \in I_1, i_2 \in I_2\}$, for all $I_1, I_2 \in \mathcal{I}(R)$.

Some examples of lattices

Example 6

The collection $\mathcal{S}(V)$ of all subspaces of a vector space V is a lattice.

Some classes of lattices

- A **bounded lattice** is an algebraic structure $(L, \vee, \wedge, 0, 1)$, such that $(L, \vee, \wedge)$ is a lattice and the constants $0, 1 \in L$ which stand respectively for the least and the greatest elements of L , satisfy the following, for all $x \in L$:
  (i) $x \wedge 1 = x$ and $x \vee 1 = 1$,
  (ii) $x \wedge 0 = 0$ and $x \vee 0 = x$
- A **complemented lattice** is a bounded lattice $(L, \vee, \wedge, 0, 1)$ in which every element has a complement, i.e., for each element $a \in L$ there exists an element $b \in L$ called its *complement* such that $a \vee b = 1$ and $a \wedge b = 0$.
- A lattice $(L, \wedge, \vee)$ is said to be **complete** if for every subset X of L , $supX$ and $infX$ exist in L .

Some classes of lattices

- A subset X of a lattice $(L, \wedge, \vee)$ is called a **sublattice** of L if X is closed under the operations of L , that is $x \wedge y \in X$ and $x \vee y \in X$, for every $x, y \in X$.
- We say that a lattice $(L, \wedge, \vee)$ is **distributive** if it satisfies one of the following conditions, for all $x, y, z \in L$

  (D1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;
  (D2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.
- A modular lattice is a lattice which satisfies one of the following equivalents conditions (modular law):

  (i) $(x \wedge y) \vee (y \wedge z) = y \wedge ((x \wedge y) \vee z)$;
  (ii) $x \leq y$ implies $x \vee (y \wedge z) = y \wedge (x \vee z)$.

Some classes of lattices (cont.)

Exercise 7

1. *Show that the conditions* $(D1)$ *and* $(D2)$ *of distributivity are equivalent.*
2. *Show that every distributive lattice is a modular lattice.*

Closure operator and interior operator

Definition 8

Let P be a Poset. A map $c : P \longrightarrow P$ is a **closure operator** *on P if the following properties hold for all $x, y \in P$:*

(i) $x \leq y$ *implies* $c(x) \leq c(y)$ (isotone)

(ii) $x \leq c(x)$ (extensive)

(iii) $c(c(x)) = c(x)$ (idempotent)

Closure operator and interior operator

Definition 8

Let P be a Poset. A map $c : P \longrightarrow P$ is a **closure operator** *on P if the following properties hold for all $x, y \in P$:*

(i) $x \leq y$ *implies* $c(x) \leq c(y)$ *(*isotone *)*

(ii) $x \leq c(x)$ *(*extensive*)*

(iii) $c(c(x)) = c(x)$ *(*idempotent*)*

If property (ii) is replaced by (ii)' : $c(x) \leq x$, then c is called an **interior operator** on P.

Closure operator and interior operator (cont.)

Let c be a closure operator on a poset P. We denote by $C := \{x \in P : c(x) = x\}$ the set of closed elements of P.

Theorem 9

Let $(L, \wedge, \vee)$ be a lattice and $c : L \to L$ a closure operator on L. Then, C is closed under arbitrary meets.

Boolean algebra

Definition 10

A **Boolean algebra** *is an algebra* $(B, \vee, \wedge, \neg, 0, 1)$ *of type* $(2, 2, 1, 0, 0)$ *which satisfies the following, for all* $x \in B$*:*

- *(i)* $(B, \vee, \wedge)$ *is a complemented distributive lattice;*
- *(ii)* $x \wedge 0 = 0$; $x \vee 1 = 1$;
- *(iii)* $x \vee \neg x = 1$; $x \wedge \neg x = 0$, *with* $\neg x$ *being the complement of* x.

Lattice morphism

Definition 11

Let L_1, L_2 be two lattices and $f : L_1 \longrightarrow L_2$ a function. Then, f is a **lattice morphism** *if $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \vee y) = f(x) \vee f(y)$, $\forall x, y \in L_1$. A lattice morphism is an isomorphism if it is bijective.*

Ideals and filters of a lattice

Let $(L, \wedge, \vee)$ be a lattice. A nonempty subset I of L is called a **lattice ideal** (or ℓ-**ideal**) of L if:

(i) $\forall x, y \in I, x \vee y \in I$;

(ii) $\forall x \in L, \forall y \in I, x \leq y \Rightarrow x \in I$.

Ideals and filters of a lattice

Let $(L, \wedge, \vee)$ be a lattice. A nonempty subset I of L is called a **lattice ideal** (or ℓ-**ideal**) of L if:

(i) $\forall x, y \in I, x \vee y \in I$;

(ii) $\forall x \in L, \forall y \in I, x \leq y \Rightarrow x \in I$.

- A subset X of L is said to be **proper** if $X \neq L$.
- A proper ideal is called **prime** if $x \wedge y \in I$ implies $x \in I$ or $y \in I$.

Ideals and filters of a lattice

Let $(L, \wedge, \vee)$ be a lattice.
A nonempty subset F of L is a **lattice filter** (**ℓ-filter**) of L if:

(i) $\forall x, y \in F, x \wedge y \in F$;

(ii) $\forall x \in F, \forall y \in L, x \leq y \Rightarrow y \in F$.

Ideals and filters of a lattice

Let $(L, \wedge, \vee)$ be a lattice.
A nonempty subset F of L is a **lattice filter** (ℓ-**filter**) of L if:

(i) $\forall x, y \in F, x \wedge y \in F$;

(ii) $\forall x \in F, \forall y \in L, x \leq y \Rightarrow y \in F$.

- A proper filter F of L is a **prime** filter if $x \vee y \in F$ implies $x \in F$ or $y \in F$.
- In lattice theory, ideals and filters come in pairs.

Ideals and filters of a lattice (cont.)

Theorem 12

Let L be a lattice. If F is a filter and I is a lattice ideal of L such that $I \cap F = \emptyset$, then there exists a prime filter P of L such that $F \subseteq P$ and $P \cap I = \emptyset$.

References

T. S. Blyth, *Lattices and ordered algebraic structures*, Springer, London,(2005).

G. Grätzer, *Lattice Theory. First Concepts and Distributive Lattices*, Courier Corporation, (2009).

Residuated Lattices and Residuated Multilattices with Applications

Pr. Célestin Lélé, CIMPA-ICTP course

June 17, 2024

Course content

Chapter 1 : Lattices
Chapter 2 : Residuated lattices
Chapter 3 : Some new classes on residuated lattices
Chapter 4 : Multilattices
Chapter 5 : Residuated multilattices
Chapter 6 : Subclasses of residuated multilattices
Chapter 7 : Applications

Chapter 2 : Residuated lattices

Chapter 2 : Residuated lattices

Residuation is a fundamental concept of ordered structures, and residuated lattices are obtained by adding a residuated monoid operation on lattices.

Table of contents

What is a Residuated lattice?

A **bounded commutative integral residuated lattice** which here is called **residuated lattice** is an algebraic structure $(A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$, satisfying the following three conditions:

(RL1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice;

(RL2) $(A, \odot, 1)$ is a commutative ordered monoid;

(RL3) For every $x, y, z \in A$, $x \leq y \rightarrow z$ iff $x \odot y \leq z$.

What is a Residuated lattice?

A **bounded commutative integral residuated lattice** which here is called **residuated lattice** is an algebraic structure $(A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$, satisfying the following three conditions:

(RL1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice;

(RL2) $(A, \odot, 1)$ is a commutative ordered monoid;

(RL3) For every $x, y, z \in A$, $x \leq y \rightarrow z$ iff $x \odot y \leq z$.

- This last property is called **residuation law**.

Some examples of Residuated lattices

Example 1

Let R be a commutative unitary ring, and $\mathcal{I}(R)$ the set of ideals of R. Then, $(\mathcal{I}(R), \wedge, \vee, \odot, \rightarrow, 0, 1 = R)$ is a residuated lattice in which $I_1 \wedge I_2 = I_1 \cap I_2$, $I_1 \vee I_2 = I_1 + I_2 = \{i_1 + i_2 : i_1 \in I_1, i_2 \in I_2\}$, $I_1 \odot I_2 = \{\sum_{k=1}^{n} f_k h_k : f_k \in I_1, h_k \in I_2, n \in \mathbb{N}^\star\}$, $I_1 \rightarrow I_2 = \{x \in R : x \cdot I_1 \subseteq I_2\}$, for all $I_1, I_2 \in \mathcal{I}(R)$.

Some examples of Residuated lattices

Example 1

Let $(B, \wedge, \vee, \neg, 0, 1)$ be a Boolean algebra. If $x \odot y = x \wedge y$ and $x \to y = \neg x \vee y$ for every $x, y \in B$, then $(B, \wedge, \vee, \odot, \to, 0, 1)$ becomes a residuated lattice.

Some examples of Residuated lattices

A **continuous t-norm** is a continuous application $T : [0,1] \times [0,1] \longrightarrow [0,1]$ such that $([0,1], T, 1)$ is a commutative ordered monoid. Each continuous t-norm generates the operation "$\rightarrow$" in the following way:

$$x \rightarrow y = max\{z \in [0;1] : T(x,z) \leq y\}.$$

There are 3 fundamental continuous t-norms: Łukasiewicz, *Gödel* and product.

Example 1

If for all $x, y \in [0,1]$ we have $x \odot y = max\{0, x+y-1\}$ and $x \rightarrow y = \min\{1, y-x+1\}$, then $([0,1], \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a residuated lattice, called the Łukasiewicz structure.

Some examples of Residuated lattices

A **continuous t-norm** is a continuous application $T : [0,1] \times [0,1] \longrightarrow [0,1]$ such that $([0,1], T, 1)$ is a commutative ordered monoid. Each continuous t-norm generates the operation "$\rightarrow$" in the following way:

$$x \rightarrow y = max\{z \in [0;1] : T(x,z) \leq y\}.$$

There are 3 fundamental continuous t-norms: Łukasiewicz, *Gödel* and product.

Example 1

The *Gödel* structure $([0,1], \vee, \wedge, \odot, \rightarrow, 0, 1)$ for which $x \odot y = min\{x, y\}$ and $x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$, for all $x, y \in [0,1]$ is a residuated lattice.

Some examples of Residuated lattices

A **continuous t-norm** is a continuous application $T : [0,1] \times [0,1] \longrightarrow [0,1]$ such that $([0,1], T, 1)$ is a commutative ordered monoid. Each continuous t-norm generates the operation "$\rightarrow$" in the following way:

$$x \rightarrow y = max\{z \in [0;1] : T(x,z) \leq y\}.$$

There are 3 fundamental continuous t-norms: Łukasiewicz, *Gödel* and product.

Example 1

If we consider $x \odot y = x \times y$ and $x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise} \end{cases}$, for all $x, y \in [0,1]$, then $([0,1], \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a residuated lattice also known as the Products structure or Gaines structure.

Some notations

Let $\mathcal{A}$ be a residuated lattice. [1]

- For any $x \in A$, we define the **negation** by $x' := x \to 0$ and for any subset X of A, $X' := \{x' : x \in X\}$.
- The set of elements of A having their negation in X, denoted $N(X)$ is defined by: $N(X) := \{x \in A : x' \in X\}$.
- For any $x \in A$ and $n \in \mathbb{N}^\star$, $x^n := \underbrace{x \odot \cdots \odot x}_{\text{n times}}$.

[1] Unless otherwise specified, $\mathcal{A}$ will always denote a residuated lattice $(A, \vee, \wedge, \odot, \to, 0, 1)$.

Some notations

Let $\mathcal{A}$ be a residuated lattice. [1]

- For any $x \in A$, we define the **negation** by $x' := x \to 0$ and for any subset X of A, $X' := \{x' : x \in X\}$.
- The set of elements of A having their negation in X, denoted $N(X)$ is defined by: $N(X) := \{x \in A : x' \in X\}$.
- For any $x \in A$ and $n \in \mathbb{N}^\star$, $x^n := \underbrace{x \odot \cdots \odot x}_{\text{n times}}$.

Remark 2

1. *If $X \subseteq Y$, then $X' \subseteq Y'$, for all $X, Y \subseteq A$.*
2. *For any nonempty subset X, Y of A, if $X \subseteq Y$, then $N(X) \subseteq N(Y)$.*

[1] Unless otherwise specified, $\mathcal{A}$ will always denote a residuated lattice $(A, \vee, \wedge, \odot, \to, 0, 1)$.

Subclasses of residuated lattices

Let $\mathcal{A}$ be a residuated lattice. Then, $\mathcal{A}$ is:

(i) A **RL- monoid** if $x \wedge y = x \odot (x \rightarrow y)$, for all $x, y \in A$ (divisibility).

(i) A **De Morgan residuated lattice** if $\mathcal{A}$ verifies the De Morgan law $(x \wedge y)' = x' \vee y'$, for all $x, y \in A$.

(ii) A **Stone algebra** if $x \wedge x'' = 1$, for all $x \in A$.

(iii) An **MTL-algebra** if it satisfies the prelinearity property $(x \rightarrow y) \vee (y \rightarrow x) = 1$, for all $x, y \in A$.

(iv) A **BL-algebra** if it is an MTL-algebra satisfying the divisibility law.

(v) **Regular** if the double negation condition $x'' = x$ holds for all x in A.

(vi) An **MV-algebra** if it is a regular BL-algebra.

(vii) A **Heyting algebra** if $\wedge = \odot$.

Subclasses of residuated lattices (cont.)

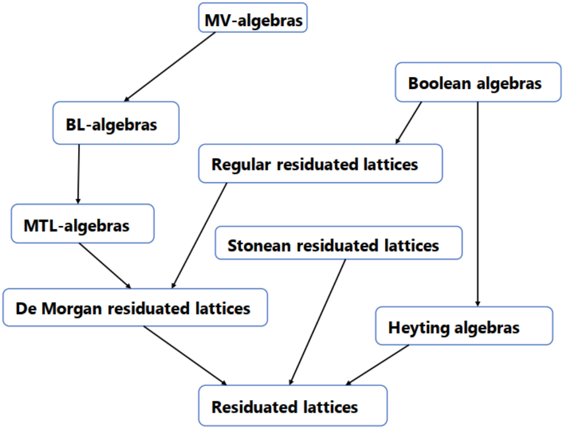


Figure 1: The interrelation between some subclasses of residuated lattices (the direction of the arrow in the diagram means "is contained in").

Useful rules of calculus in resisuated lattices

For all $x, y, z \in A$, $\{x_\gamma : \gamma \in \Gamma \neq \emptyset\} \subseteq A$ where Γ is a finite subset of nonzero integers , we have the following rules:

(P1) $x \leq y$ iff $x \to y = 1, \quad x \odot y \leq x \wedge y, \quad x \odot y \leq x \to y;$

(P2) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z);$

(P3) $1 \to x = x, \quad x \to x = 1, \quad x \to 1 = 1, \quad x \leq y \to x, \quad 0' = 1$ and $1' = 0;$

(P4) If $x \leq y$, then $y \to z \leq x \to z$, $z \to x \leq z \to y, \quad x \odot z \leq y \odot z, \quad y' \leq x';$

(P5) $(x \odot y)' = x \to y' = y \to x' = x'' \to y', x \odot (x \to y) \leq y, x \leq (x \to y) \to y;$

(P6) $x \to y \leq y' \to x', \quad x \leq x'', \quad x''' = x';$

(P7) $x \odot x' = 0, \quad x \odot y = 0$ iff $x \leq y';$

(P8) $(x \vee y)' = x' \wedge y', \quad (x \wedge y)' \geq x' \vee y', \quad (x' \wedge y')'' = x' \wedge y', \quad (x'' \wedge y'')'' = x'' \wedge y'';$

Useful rules of calculus in resisuated lattices

For all $x, y, z \in A$, $\{x_\gamma : \gamma \in \Gamma \neq \emptyset\} \subseteq A$ where Γ is a finite subset of nonzero integers , we have the following rules:

(P9) $y \odot (\bigvee_{\gamma\in\Gamma} x_\gamma) = \bigvee_{\gamma\in\Gamma}(y \odot x_\gamma)$, $y \odot (\bigwedge_{\gamma\in\Gamma} x_\gamma) \leq \bigwedge_{\gamma\in\Gamma}(x_\gamma \odot y)$, $x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$ and hence $(x \vee y)^{mn} \leq x^m \vee y^n$, for every $n, m \geq 1$;

(P10) $y \to (\bigvee_{\gamma\in\Gamma} x_\gamma) = \bigwedge_{\gamma\in\Gamma}(y \to x_\gamma)$, $(\bigvee_{\gamma\in\Gamma} x_\gamma) \to z = \bigwedge_{\gamma\in\Gamma}(x_\gamma \to z)$, $(\bigwedge_{\gamma\in\Gamma} x_\gamma) \to z \geq \bigvee_{\gamma\in\Gamma}(x_\gamma \to z)$, $y \to (\bigvee_{\gamma\in\Gamma} x_\gamma) \geq \bigvee_{\gamma\in\Gamma}(y \to x_\gamma)$;

(P11) $x' \odot y' \leq (x \odot y)'$, $x'' \odot y'' \leq (x \odot y)''$, $x' \odot y' \leq (x' \to y)'$ and $x, y \leq (x' \odot y')'$;

(P12) $x \vee y = 1$ implies $x \odot y = x \wedge y$ and $x^n \vee y^n = 1$, for every $n \geq 1$.

Useful rules of calculus in resisuated lattices (cont. 1)

The operation $\oslash$ defined for every $x, y \in A$ by $x \oslash y := x' \to y$ which is neither associative nor commutative, but compatible with the order is called **pseudo-addition**.

Useful rules of calculus in resisuated lattices (cont. 1)

The operation $\oslash$ defined for every $x, y \in A$ by $x \oslash y := x' \to y$ which is neither associative nor commutative, but compatible with the order is called **pseudo-addition**.

Remark 3

We easily see that the operation $\oslash$ verifies the following, for every $x, y, z \in A$.

(i) $x \oslash (y \oslash z) = y \oslash (x \oslash z)$;

(ii) $x \oslash (y \wedge z) = (x \oslash y) \wedge (x \oslash z)$;

(iii) $(y \wedge z) \oslash x \leq (y \oslash x) \wedge (z \oslash x)$;

(iv) $x \oslash (y \vee z) \geq (x \oslash y) \vee (x \oslash z)$;

(v) $(y \vee z) \oslash x \geq (y \oslash x) \vee (z \oslash x)$.

Useful rules of calculus in resisuated lattices (cont. 2)

The operation $\oplus$ defined by $x \oplus y := (x' \odot y')'$, for all $x, y \in A$, is commutative, associative and compatible with the order, properties that makes it a more interesting operation than the pseudo-addition.

Useful rules of calculus in resisuated lattices (cont. 2)

The operation $\oplus$ defined by $x \oplus y := (x' \odot y')'$, for all $x, y \in A$, is commutative, associative and compatible with the order, properties that makes it a more interesting operation than the pseudo-addition.

For any $x \in A$, $nx := \underbrace{x \oplus \cdots \oplus x}_{\text{n times}}$, $n \in \mathbb{N}^\star$. We have :

(P13) $[(x')^n]' = nx$, $x \wedge (ny) \leq n(x'' \wedge y'')$ and $(mx) \wedge (ny) = (mn)(x'' \wedge y'')$ for every $m, n \geq 2$.

Homomorphism of resisuated lattices

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two residuated lattices. A function $f : A_1 \longrightarrow A_2$ is called **homomorphism of residuated lattices** if it is a homomorphism of lattices and : $f(0) = 0$, $f(x \odot y) = f(x) \odot f(y)$, $f(x \rightarrow y) = f(x) \rightarrow f(y)$, for all $x, y \in A_1$.

Homomorphism of resisuated lattices

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two residuated lattices. A function $f : A_1 \longrightarrow A_2$ is called **homomorphism of residuated lattices** if it is a homomorphism of lattices and : $f(0) = 0$, $f(x \odot y) = f(x) \odot f(y)$, $f(x \to y) = f(x) \to f(y)$, for all $x, y \in A_1$.

- If $f : A_1 \longrightarrow A_2$ is an homomorphism of residuated lattices, then $f(1) = 1$.

Table of contents

Filters

Definition 4

Let $\mathcal{A}$ be a residuated lattice. A nonempty subset F of A is called a **filter** *of $\mathcal{A}$ if it satisfies the following conditions:*

(i) $\forall x, y \in F, x \odot y \in F$;

(ii) $\forall\ x \in F, \forall y \in A$, *if* $x \leq y$, *then* $y \in F$.

- We denote by $Fil(\mathcal{A})$ the set of all filters of $\mathcal{A}$.

Deductive system

Definition 5

A **deductive system** *of a residuated lattice $\mathcal{A}$ is a nonempty subset F of A containing 1 such that for all $x, y \in A$, $x \to y \in F$ and $x \in F$ imply $y \in F$.*

Filters and deductive systems

Exercise 6

Show that in a residuated lattice, filters and deductive systems coincide.

Exercise 6

Show that in a residuated lattice, filters and deductive systems coincide.

Solution : Let $F \subseteq A$ be a non void subset of A.
$\Rightarrow$) Assume that F is a filter;
i) For every $x \in A$, $x \leq 1$. More precisely, for every $x \in F$, $x \leq 1$. Since F is a filter, it yields that $1 \in F$.
ii) Let $x \in F$, and $x \to y \in F$. Then, $x \odot (x \to y) \in F$, since F is a filter.
But we know that $x \odot (x \to y) \leq y$.
Hence $y \in F$, implying that F is a deductive system.

Exercise 6

Show that in a residuated lattice, filters and deductive systems coincide.

Solution : Let $F \subseteq A$ be a non void subset of A.
$\Leftarrow$) Conversely, let's show that F is a filter:
i) Let $x, y \in F$
We know that :

$$\begin{aligned} x \to [y \to (x \odot y)] &= (x \odot y) \to (x \odot y). \\ &= 1 \in F \textit{ by hypothesis}. \end{aligned}$$

That is x and $x \to [y \to (x \odot y)]$ are in F, which implies that $y \to (x \odot y) \in F$.
We also have $y \in F$ and $y \to (x \odot y) \in F$;
Thus, $x \odot y \in F$.
ii) Let $x \in F$ and $x \leq y$, for every $y \in A$. Since $x \leq y$, then $x \to y = 1 \in F$. That is $x \in F$ and $x \to y \in F$. Therefore, $y \in F$. And we have the result.

Properties of filters

- For any homomorphism $f : A_1 \longrightarrow A_2$ and any filter F of A_2, $f^{-1}(F)$ is a filter of A_1.
- If F is a filter of $\mathcal{A}$, then F is also a lattice filter of $\mathcal{A}$.

Prime filters and maximal filters

- A proper filter P of $\mathcal{A}$ is called a **prime filter** if $x \vee y \in P$ implies $x \in P$ or $y \in P$, for all $x, y \in A$.
- A filter M of $\mathcal{A}$ is **maximal** if it is a maximal element of the set of all proper filters of $\mathcal{A}$.

We denote by $Spec(\mathcal{A})$ the set of prime filters of $\mathcal{A}$ and by $Max(\mathcal{A})$ the set of maximal filters.

Prime filters and maximal filters

- A proper filter P of $\mathcal{A}$ is called a **prime filter** if $x \vee y \in P$ implies $x \in P$ or $y \in P$, for all $x, y \in A$.
- A filter M of $\mathcal{A}$ is **maximal** if it is a maximal element of the set of all proper filters of $\mathcal{A}$.

We denote by $Spec(\mathcal{A})$ the set of prime filters of $\mathcal{A}$ and by $Max(\mathcal{A})$ the set of maximal filters.

Exercise 7

Show that every maximal filter is a prime filter.

Some types of filters

Let F be a filter of $\mathcal{A}$, $n \in \mathbb{N}^\star$. Then,

(i) F is called an **n-fold boolean** filter if for all $x \in A$, $x \vee (x^n)' \in F$.

(ii) F is said to be **n-fold implicative** if for all $x, y, z \in A$, $x^n \to (y \to z) \in F$ and $x^n \to y \in F$ imply $x^n \to z \in F$.

(iii) F is an **n-fold obstinate** filter if for all $x \in A, x \notin F$ implies $(x^n)' \in F$.

Some types of filters

Let F be a filter of $\mathcal{A}$, $n \in \mathbb{N}^\star$. Then,

(i) F is called an **n-fold boolean** filter if for all $x \in A$, $x \vee (x^n)' \in F$.

(ii) F is said to be **n-fold implicative** if for all $x, y, z \in A$, $x^n \to (y \to z) \in F$ and $x^n \to y \in F$ imply $x^n \to z \in F$.

(iii) F is an **n-fold obstinate** filter if for all $x \in A, x \notin F$ implies $(x^n)' \in F$.

Proposition 8

Let F be a filter of $\mathcal{A}$, $n \in \mathbb{N}^\star$. Then, the following are equivalent:

(i) *F is maximal and n-fold implicative.*

(ii) *F is maximal and n-fold boolean.*

(iii) *F is prime and n-fold boolean.*

(iv) *F is n-fold obstinate.*

Generated filter

For any nonempty subset X of A, the least filter of $\mathcal{A}$ containing X (called the filter generated by X) will be denoted $[X)$, and for all $x \in A$, $[\{x\})$ will be denoted $[x)$.

Generated filter

For any nonempty subset X of A, the least filter of $\mathcal{A}$ containing X (called the filter generated by X) will be denoted $[X)$, and for all $x \in A$, $[\{x\})$ will be denoted $[x)$.

Proposition 9

(i) $[X) := \{a \in A : x_1 \odot ... \odot x_n \leq a,$ *for some* $n \in \mathbb{N}^\star$ *and* $x_1, x_2, ..., x_n \in X\}$. *Particularly,* $[x) = \{a \in A : x^n \leq a,$ *for some* $n \in \mathbb{N}^\star\}$ *is called principal filter.*

(ii) $(Fil(\mathcal{A}), \wedge, \vee, \{1\}, A)$ *is a complete distributive lattice in which for any nonempty family* $\mathcal{F}$ *of* $Fil(\mathcal{A})$, $\wedge(\mathcal{F}) := \cap\{F : F \in \mathcal{F}\}$, $\vee(\mathcal{F}) := [\cup\{F : F \in Fil(\mathcal{A})\})$.

Co-annihilator

Definition 10

Let X be a nonempty subset of A. The **co-annihilator** *of X is the filter* ${}^{\top}X = \{x \in A : x \vee y = 1, \forall y \in X\}$.

We denote ${}^{\top}\{x\}$ by ${}^{\top}x$, and $CoAnn(\mathcal{A}) := \{ {}^{\top}X : X \subseteq A\}$.

Co-annihilator

Definition 10

Let X be a nonempty subset of A. The **co-annihilator** *of X is the filter* ${}^{\top}X = \{x \in A : x \vee y = 1, \forall y \in X\}$.

We denote ${}^{\top}\{x\}$ by ${}^{\top}x$, and $CoAnn(\mathcal{A}) := \{ {}^{\top}X : X \subseteq A\}$.

Proposition 11

Let $\mathcal{A}$ be a residuated lattice. Then, $(CoAnn(\mathcal{A}), \vee^{\top}, \cap, {}^{\top}, 1, A)$ is a complete boolean algebra in which $F \vee^{\top} G = (F^{\top} \cap g^{\top})^{\top}$, for all $F, G \in CoAnn(\mathcal{A})$.

Table of contents

Ideals

Definition 12

A nonempty subset I of A is called an **ideal** *of $\mathcal{A}$ if:*

(I1) *For every $x, y \in I$, $x \oslash y \in I$;*

(I2) *For every $x, y \in A$, if $x \leq y$ and $y \in I$, then $x \in I$.*

We denote by $\mathcal{I}(\mathcal{A})$ the set of ideals of $\mathcal{A}$.

Ideals

Definition 12

A nonempty subset I of A is called an **ideal** *of $\mathcal{A}$ if:*

(I1) *For every $x, y \in I$, $x \oslash y \in I$;*

(I2) *For every $x, y \in A$, if $x \leq y$ and $y \in I$, then $x \in I$.*

We denote by $\mathcal{I}(\mathcal{A})$ the set of ideals of $\mathcal{A}$.

- The notions of ideal and filter are not dual in (non regular) residuated lattices, since for an ideal I, the set $I' = \{x' : x \in I\}$ is not always a filter, and $F' = \{x' : x \in F\}$ is not necessarily an ideal for a given filter F.

characterizations of ideals

Proposition 13

Let $\mathcal{A}$ be a residuated lattice and I a nonempty subset of A. Then, the following are equivalent:

1. *I is an ideal of $\mathcal{A}$;*
2. (i) $0 \in I$
   (ii) *For every $x, y \in A$, if $x' \oslash y \in I$ and $x \in I$, then $y \in I$;*
3. (i) $0 \in I$
   (ii) *For every $x, y \in A$, if $(x' \to y')' \in I$ and $x \in I$, then $y \in I$;*
4. (i) *For every $x, y \in I$, $x \oslash y \in I$,*
   (ii) *For every $x, y \in A$, if $x \vee y \in I$, then $x \in I$ and $y \in I$;*
5. (i) *For every $x, y \in I$, $x \oslash y \in I$,*
   (ii) *For every $x, y \in A$, if $x \in I$, then $x \wedge y \in I$.*
6. (i) *For every $x, y \in I$, $x \oplus y \in I$;*
   (ii) *For every $x, y \in A$, if $x \leq y$ and $y \in I$, then $x \in I$.*

Some properties of ideals

- $\{0\}$ and A are trivial ideals of $\mathcal{A}$, and each ideal of $\mathcal{A}$ contains 0.
- Let I be an ideal of $\mathcal{A}$. Then, $x \in I$ iff $x'' \in I$, for every $x \in A$.
- Ideals of $\mathcal{A}$ are lattice ideals of $\mathcal{A}$.

Some types of ideals

Definition 14

Let P be a proper ideal of $\mathcal{A}$. Then,

(i) *P is a* **maximal ideal** *of $\mathcal{A}$ if it is not contained in any other proper ideal of $\mathcal{A}$.*

(ii) *P is called a* **prime ideal** *of $\mathcal{A}$ if P is a prime element of $(\mathcal{I}(\mathcal{A}), \subseteq)$, that is, if I, J are ideals of $\mathcal{A}$ and $I \cap J \subseteq P$, then $I \subseteq P$ or $J \subseteq P$.*

(iii) *P is a* **prime ideal of the second kind** *of $\mathcal{A}$ if for every $x, y \in A$, $x \wedge y \in P$ implies $x \in I$ or $y \in P$.*

(iv) *We say that P is a* **prime ideal of third kind** *of $\mathcal{A}$ if for all $x, y \in A$, $(x \rightarrow y)' \in P$ or $(y \rightarrow x)' \in P$.*

(v) *A prime ideal P which is minimal in the poset of prime ideals containing an ideal I is called a* **minimal prime ideal belonging to** *I. A minimal prime ideal belonging to $\{0\}$ is called* **minimal prime ideal**. *In other words, P is a* **minimal prime ideal** *if P is prime, and for every prime ideal Q, if $Q \subseteq P$, then $P = Q$.*

Some types of ideals (cont.)

- Every prime ideal of third kind of $\mathcal{A}$ is a prime ideal of second kind of $\mathcal{A}$.
- Every prime ideal of second kind of $\mathcal{A}$ is also a prime ideal of $\mathcal{A}$.
- If $\mathcal{A}$ is a De Morgan residuated lattice, then a prime ideal of $\mathcal{A}$ is a prime ideal of second kind of $\mathcal{A}$ (or a prime ideal of third kind of $\mathcal{A}$).

Some types of ideals (cont.)

- Every prime ideal of third kind of $\mathcal{A}$ is a prime ideal of second kind of $\mathcal{A}$.
- Every prime ideal of second kind of $\mathcal{A}$ is also a prime ideal of $\mathcal{A}$.
- If $\mathcal{A}$ is a De Morgan residuated lattice, then a prime ideal of $\mathcal{A}$ is a prime ideal of second kind of $\mathcal{A}$ (or a prime ideal of third kind of $\mathcal{A}$).

For a residuated lattice $\mathcal{A}$, we denote by $Max_{Id}(\mathcal{A})$, $Spec_{Id}(\mathcal{A})$, and by $Min_{Id}(\mathcal{A})$ the set of maximal ideals of $\mathcal{A}$, the set of all prime ideals of $\mathcal{A}$, and the set of minimal prime ideals of $\mathcal{A}$, respectively.

- Note that $Max_{Id}(\mathcal{A}) \subseteq Spec_{Id}(\mathcal{A})$, and $Min_{Id}(\mathcal{A}) \subseteq Spec_{Id}(\mathcal{A})$.

Some descriptions of the maximum ideals

Proposition 15

Let M be a proper ideal of $\mathcal{A}$. Then, the following are equivalent:

(i) $M \in Max_{Id}(\mathcal{A})$.

(ii) *For any* $x \in A$, $x \notin M$ *iff* $(nx)' \in M$, *for some* $n \in \mathbb{N}^\star$.

(iii) *For all* $x \notin M$ *there is* $y \in M$ *and* $n \in \mathbb{N}^\star$ *such that* $y \oplus (nx) = 1$.

Some descriptions of the maximum ideals

Proposition 15

Let M be a proper ideal of $\mathcal{A}$. Then, the following are equivalent:

(i) $M \in Max_{Id}(\mathcal{A})$.

(ii) *For any* $x \in A$, $x \notin M$ *iff* $(nx)' \in M$, *for some* $n \in \mathbb{N}^{\star}$.

(iii) *For all* $x \notin M$ *there is* $y \in M$ *and* $n \in \mathbb{N}^{\star}$ *such that* $y \oplus (nx) = 1$.

- It follows from Zorn's lemma that every proper ideal of a residuated lattice can be extended to a maximal ideal.

Some characterizations of prime ideals

Proposition 16

Let P be a proper ideal of $\mathcal{A}$. Then, the following are equivalent:

(i) *P is prime.*

(ii) *$x'' \wedge y'' \in P$ implies $x \in P$ or $y \in P$, for all $x, y \in A$.*

(iii) *If $I, J \in \mathcal{I}(\mathcal{A})$ and $I \cap J = P$, then $I = P$ or $J = P$.*

Theorem 17

(Prime ideal theorem) Let $\mathcal{A}$ be a residuated lattice. If I is an ideal and F is a lattice filter of $\mathcal{A}$ such that $I \cap F = \emptyset$, then there exists a prime ideal P of $\mathcal{A}$ such that $I \subseteq P$ and $P \cap F = \emptyset$.

Some characterizations of prime ideals (cont.)

As a direct consequence of the prime ideal theorem, any proper ideal I of $\mathcal{A}$ can be extended to a prime ideal, that is $I = \cap\{P \in spec_{Id}(\mathcal{A}) : I \subseteq P\}$. Subsequently, we have the following:

Some characterizations of prime ideals (cont.)

As a direct consequence of the prime ideal theorem, any proper ideal I of $\mathcal{A}$ can be extended to a prime ideal, that is $I = \cap\{P \in spec_{Id}(\mathcal{A}) : I \subseteq P\}$. Subsequently, we have the following:

Proposition 18

For every ideal I of $\mathcal{A}$ and $x \in A \setminus I$, there is a minimal prime ideal P such that $I \subseteq P$ and $x \notin P$. Singularly, for every $x \in A$, there exists a minimal prime ideal P such that $x \notin P$, whenever $x \neq 0$.

Ideal generated

For any nonempty subset X of A, the least ideal of $\mathcal{A}$ containing X (called the ideal generated by X) will be denoted $\langle X \rangle$, and for all $x \in A$, $\langle \{x\} \rangle$ will be denoted $\langle x \rangle$.

Ideal generated

For any nonempty subset X of A, the least ideal of $\mathcal{A}$ containing X (called the ideal generated by X) will be denoted $\langle X \rangle$, and for all $x \in A$, $\langle \{x\} \rangle$ will be denoted $\langle x \rangle$.

Proposition 19

Let $\mathcal{A}$ be a residuated lattice, and $x \in A$. Then

(i) $\langle X \rangle := \{a \in A : a \leq x_1 \oplus \ldots \oplus x_n,$ *for some* $n \in \mathbb{N}^\star$ *and* $x_1, x_2, \ldots, x_n \in X\}$. *Particularly,* $\langle x \rangle = \{a \in A : a \leq nx,$ *for some* $n \in \mathbb{N}^\star\}$, *called principal ideal.*

(ii) $(\mathcal{I}(\mathcal{A}), \subseteq)$ *is a complete Heyting algebra where for any nonempty family* $\mathcal{F} \in \mathcal{I}(\mathcal{A})$, $inf(\mathcal{F}) := \cap\{I : I \in \mathcal{F}\}$, $sup(\mathcal{F}) := \langle \cup\{I : I \in \mathcal{F}\} \rangle$, $I \to J := \{x \in A : \langle x \rangle \cap I \subseteq J\}$, *for any* $I, J \in \mathcal{I}(\mathcal{A})$ *and* $I' = I \to \{0\}$.

Relations between ideals and filters

Proposition 20

Let F be a filter and I an ideal of $\mathcal{A}$. Then,

(N1) $I = N(N(I))$;

(N2) $F \subseteq N(N(F))$;

(N3) $N(I)$ *is a filter and* $I' \subseteq N(I)$;

(N4) $N(F)$ *is an ideal generated by* F' ;

(N5) $N(F) = N(N(N(F)))$.

Table of contents

Congruence

It is well known that an equivalence relation $\sim$ on $\mathcal{A}$ is called a $\{\delta\}$-*congruence* relation on $\mathcal{A}$ with $\delta \in \{\wedge, \vee, \odot, \rightarrow\}$ if it preserves the operator δ, that is, $(a, b) \in \ \sim$ and $(c, d) \in \ \sim$ imply $(a \ \delta \ c, b \ \delta \ d) \in \ \sim$.
We say that an equivalence relation $\sim$ of $\mathcal{A}$ is a **congruence** relation on $\mathcal{A}$ if $\sim$ preserves all operators of $\mathcal{A}$. We will denote by $Con(\mathcal{A})$ the set of congruence relations on $\mathcal{A}$, and for all $a, b \in A$ we will denote $(a, b) \in \ \sim$ by $a \sim b$.

Congruence

It is well known that an equivalence relation $\sim$ on $\mathcal{A}$ is called a $\{\delta\}$-*congruence* relation on $\mathcal{A}$ with $\delta \in \{\wedge, \vee, \odot, \rightarrow\}$ if it preserves the operator δ, that is, $(a, b) \in \ \sim$ and $(c, d) \in \ \sim$ imply $(a \ \delta \ c, b \ \delta \ d) \in \ \sim$.
We say that an equivalence relation $\sim$ of $\mathcal{A}$ is a **congruence** relation on $\mathcal{A}$ if $\sim$ preserves all operators of $\mathcal{A}$. We will denote by $Con(\mathcal{A})$ the set of congruence relations on $\mathcal{A}$, and for all $a, b \in A$ we will denote $(a, b) \in \ \sim$ by $a \sim b$.

Theorem 21

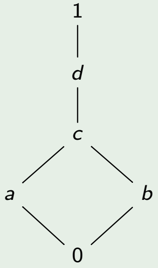
Let $\sim$ be a congruence relation on $\mathcal{A}$, and F a filter of $\mathcal{A}$. Then, there is an isomorphism between $Con(\mathcal{A})$ and $Fil(\mathcal{A})$.

Examples of congruences

Example 22

Let $A = \{0, a, b, c, d, 1\}$ be a set with the lattice structure presented in figure to the right, the operations $\rightarrow$ and $\odot$ defined in tables.

- $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice.
- $\mathcal{I}(\mathcal{A}) = \{\{0\}, A\}$, and $Fil(\mathcal{A}) = \{\{1\}, F = \{d, 1\}, A\}$.
- The 3 relations: $\sim_{\{1\}} = \{(0,0), (a,a), (b,b), (c,c), (d,d), (1,1)\}$, $\sim_F = \{(0,0), (a,a), (b,b), (c,c), (d,d), (1,1), (d,1)\}$ and $\sim_A = A \times A$ are congruence relations on $\mathcal{A}$.



$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	c	1	1	1
b	c	c	1	1	1	1
c	c	c	c	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	a	a
b	0	0	0	0	b	b
c	0	0	0	0	c	c
d	0	a	b	c	d	d
1	0	a	b	c	d	1

Ideals, filters and congruences

- Ideals and filters are not dual notions in residuated lattices
- Some residuated lattices have more filters than ideals (the previous example).
- In general, there cannot be a bijection between $Con(\mathcal{A})$ and $\mathcal{I}(\mathcal{A})$.
- $Con(\mathcal{A}) \cong Fil(\mathcal{A}) \cong \mathcal{I}(\mathcal{A})$ whenever $\mathcal{A}$ is regular.

References

- D. Busneag, D. Piciu, and A. M. Dina, *Ideals in residuated lattices. Carpathian Journal of mathematics*, 37(1)(2021), 53-63.
- J.A. Goguen, *L-fuzzy sets*, Journal of Mathematical Analysis and Applications, 18 (1967), 145-174.
- D. Piciu, *Algebras of fuzzy logic*, Editura Universitaria Craiova, Craiova. Romania (2007).
- E. Turunen, *Mathematics Behind Fuzzy Logic*, Advances in Intelligent and Soft Computing, Heidelberg: Physica-Verlag, (1999).
- B. Van Gasse, G. Deschrijver and C. Cornelis, *Filters of residuated lattices and triangle algebras*, Information Sciences, 180 (16) (2010), 3006-3020.
- M. Ward and R. P. Dilworth, *Residuated lattices*, Proceedings of the National Academy of Sciences 24 (3) (1938), 162-164.

Residuated Lattices and Residuated Multilattices with Applications

Pr. Célestin Lélé, CIMPA-ICTP course

June 17-20, 2024

CIMPA

Chapter 3 : Some new classes on residuated lattices

Main objectives

This chapter studies and gives new characterizations of semi-idempotent and cover related classes of algebras. In addition, it also introduces a new class of residuated lattices called semi-prelinear residuated lattices, a class that is shown to include all MTL-algebras. Finally, the number of semi-prelinear residuated lattices, semi-idempotent residuated lattices, semi-divisible residuated lattices, De Morgan residuated lattices and Stonean residuated lattices up to order 12 is computed.

Table of contents

Semi-idempotent residuated lattice

- A residuated lattice $\mathcal{A}$ is called **idempotent** if it satisfies: $x \odot x = x$ for all $x \in A$.

Semi-idempotent residuated lattice

- A residuated lattice $\mathcal{A}$ is called **idempotent** if it satisfies: $x \odot x = x$ for all $x \in A$.

It is a customary approach to weaken the definition of an equational class $\mathcal{C}$ by replacing elements in the equations by their complements and calling the resulting class semi-$\mathcal{C}$.

Semi-idempotent residuated lattice

- A residuated lattice $\mathcal{A}$ is called **idempotent** if it satisfies: $x \odot x = x$ for all $x \in A$.

It is a customary approach to weaken the definition of an equational class $\mathcal{C}$ by replacing elements in the equations by their complements and calling the resulting class semi-$\mathcal{C}$.

Definition 1

A residuated lattice $\mathcal{A}$ is called *semi-idempotent* if it satisfies: $x \in A$, $(x' \odot x')' = x''$ for all $x \in A$.

Semi-idempotent residuated lattice

- A residuated lattice $\mathcal{A}$ is called **idempotent** if it satisfies: $x \odot x = x$ for all $x \in A$.

It is a customary approach to weaken the definition of an equational class $\mathcal{C}$ by replacing elements in the equations by their complements and calling the resulting class semi-$\mathcal{C}$.

Definition 1

A residuated lattice $\mathcal{A}$ is called *semi-idempotent* if it satisfies: $x \in A$, $(x' \odot x')' = x''$ for all $x \in A$.

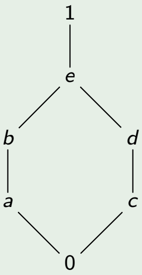
- Every idempotent residuated lattice is semi-itempotent.

Example of a semi-idempotent residuated lattice

Example 2

Let $A = \{0, a, b, c, d, e, 1\}$ be a set with the lattice structure presented in figure to the right, the operations $\rightarrow$ and $\odot$ defined in tables.

- $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a semi-idempotent residuated lattice which is not idempotent.



→	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	d	1	1	d	d	1	1
b	d	e	1	d	d	1	1
c	b	b	b	1	1	1	1
d	b	b	b	e	1	1	1
e	0	b	b	d	d	1	1
1	0	a	b	c	d	e	1

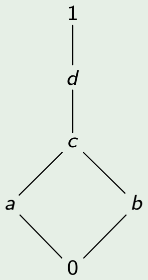
⊙	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	a	0	0	a	a
b	0	a	a	0	0	a	b
c	0	0	0	c	c	c	c
d	0	0	0	c	c	c	d
e	0	a	a	c	c	e	e
1	0	a	b	c	d	e	1

Example of a semi-idempotent residuated lattice

Example 2

Let $A = \{0, a, b, c, d, 1\}$ be a set with the lattice structure presented in figure to the right, the operations $\rightarrow$ and $\odot$ defined in tables.

- $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice which is neither idempotent nor semi-idempotent.



$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	1	1	1
b	c	c	1	1	1	1
c	b	c	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	a	a
b	0	0	0	0	b	b
c	0	a	0	a	c	c
d	0	a	b	c	d	d
1	0	a	b	c	d	1

Some characterizations of semi-idempotency

Proposition 1

Given a residuated lattice A, the following assertions are equivalent:

(i) *A is semi-idempotent;*

(ii) *For every $x \in A$, $(x \odot x)' = x'$;*

(iii) *For every $x, y \in A$, $(x' \to y') \to x'' = y' \to x'' = (x \vee y)''$;*

(iv) *For every $x, y \in A$, $(x \wedge y)' = (x \odot y)'$;*

(v) *A is pseudocomplemented[a];*

(vi) *For every $x \in A$, $x \to x' = x'$.*

[a] A resisuated lattice is called *pseudocomplemented* if it satisfies the property : for all $x \in A$, $x \wedge x' = 0$.

Some properties of the semi-idempotency

Proposition 2

Any semi-idempotent and regular residuated lattice is idempotent.

Proof.

Let $\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ be a semi-idempotent residuated lattice. Suppose that $\mathcal{A}$ satisfies the double negation. Let $x \in M$; we have $((x')' \odot (x')')' = (x')''$ because $\mathcal{A}$ is semi-idempotent; i.e., $(x'' \odot x'')' = x''' = x'$, i.e., $(x'' \odot x'')'' = x''$. Since $\mathcal{A}$ satisfies the double negation, we have $x \odot x = x$, so $\mathcal{A}$ is idempotent. □

Some properties of the semi-idempotency

Proposition 2

Any idempotent and regular residuated lattice is a Boolean algebra.

Proof.

Let $\mathcal{A}$ be an idempotent residuated lattice. Suppose that $\mathcal{A}$ satisfies the double negation. Let $x, y \in A$. We have, $x' \vee y = (x' \vee y)'' = (x'' \wedge y')' = (x \wedge y')' = (x \odot y')'$ because $\mathcal{A}$ is idempotent. Then, $x' \vee y = x \to y'' = x \to y$. Thus $x \to y = x' \vee y$. Moreover, since $\mathcal{A}$ is idempotent, we have $x \odot y = x \wedge y$, therefore, $\mathcal{A}$ is a Boolean algebra. □

Some properties of the semi-idempotency

Proposition 2

Any semi-idempotent and regular residuated lattice is idempotent.

Proposition 3

Any idempotent and regular residuated lattice is a Boolean algebra.

- The only semi-idempotent and regular residuated lattices are the Boolean algebras.

Some properties of the semi-idempotency (cont.)

- A residuated lattice $\mathcal{A}$ is called **divisible** if it satisfies the divisibility axiom : for all $x, y \in A$, $x \odot (x \to y) = x \wedge y$;
- A residuated lattice $\mathcal{A}$ is called **semi-divisible** if it satisfies the divisibility axiom : for all $x, y \in A$, $[x' \odot (x' \to y')]' = (x' \wedge y')'$.

Some properties of the semi-idempotency (cont.)

- A residuated lattice $\mathcal{A}$ is called **divisible** if it satisfies the divisibility axiom : for all $x, y \in A$, $x \odot (x \to y) = x \wedge y$;
- A residuated lattice $\mathcal{A}$ is called **semi-divisible** if it satisfies the divisibility axiom : for all $x, y \in A$, $[x' \odot (x' \to y')]' = (x' \wedge y')'$.

Proposition 4

Every semi-idempotent residuated lattice is semi-divisible.

Some properties of the semi-idempotency (cont.)

- A residuated lattice $\mathcal{A}$ is called **divisible** if it satisfies the divisibility axiom : for all $x, y \in A$, $x \odot (x \to y) = x \wedge y$;
- A residuated lattice $\mathcal{A}$ is called **semi-divisible** if it satisfies the divisibility axiom : for all $x, y \in A$, $[x' \odot (x' \to y')]' = (x' \wedge y')'$.

Proposition 4

Every semi-idempotent residuated lattice is semi-divisible.

Exercise 3

Prove Proposition 4

Table of contents

Semi-prelinearity in residuated lattices

Definition 4

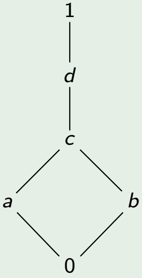
A *semi-prelinear residuated lattice* is a bounded residuated lattice $\mathcal{A}$ such that the *semi-prelinearity* equation $(x' \to y') \vee (y' \to x') = 1$ holds for all $x, y \in A$.

Example of a semi-prelinear residuated lattice

Example 5

Let $A = \{0, a, b, c, d, 1\}$ be a set with the lattice structure presented in figure to the right, the operations $\rightarrow$ and $\odot$ defined in tables.

- $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a semi-prelinear residuated lattice which is not an MTL-algebra, since $(a \rightarrow b) \vee (b \rightarrow a) = b \vee c = c \neq 1$.



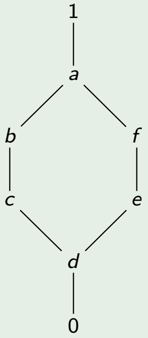
$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	1	1	1
b	c	c	1	1	1	1
c	b	c	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	a	a
b	0	0	0	0	b	b
c	0	a	0	a	c	c
d	0	a	b	c	d	d
1	0	a	b	c	d	1

Example 5

Let $A = \{0, a, b, c, d, e, f, 1\}$ be a set with the lattice structure presented in figure to the right, the operations $\rightarrow$ and $\odot$ defined in tables.

- $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice.
- We have $(b \rightarrow e) \vee (e \rightarrow b) = f \vee a = a \neq 1$ and $(b' \rightarrow e') \vee (e' \rightarrow b') = (e \rightarrow b) \vee (b \rightarrow e) \neq 1$.
- $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is neither a MTL-algebra nor a semi-prelinear residuated lattice.



$\rightarrow$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	a	a	f	f	f	1
b	e	1	1	a	f	f	f	1
c	f	1	1	1	f	f	f	1
d	a	1	1	1	1	1	1	1
e	b	1	a	a	a	1	1	1
f	c	1	a	a	a	a	1	1
1	0	a	b	c	d	e	f	1

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	c	c	c	0	d	d	a
b	0	c	c	c	0	0	d	b
c	0	c	c	c	0	0	0	c
d	0	0	0	0	0	0	0	d
e	0	d	0	0	0	d	d	e
f	0	d	d	0	0	d	d	f
1	0	a	b	c	d	e	f	1

Some properties of the semi-preliearity

Proposition 5

Every MTL-algebra is a semi-prelinear residuated lattice.

Proof.

Let $\mathcal{A}$ be a MTL-algebra. Let $x, y \in A$, we have $x', y' \in A$, then $(x' \to y') \vee (y' \to x') = 1$, because $\mathcal{A}$ satisfies the prelinearity property. Thus $\mathcal{A}$ is a semi-prelinear residuated lattice. □

Proposition 6

Every semi-prelinear and regular residuated lattice is an MTL-algebra.

Proof.

Let $\mathcal{A}$ be a residuated lattice. Suppose that $\mathcal{A}$ is a semi-prelinear residuated lattice and verifies the double negation. For all $x, y \in A$, we have $(x' \to y') \vee (y' \to x') = 1$. Then by (P8), $(y'' \to x'') \vee (x'' \to y'') = 1$. Therefore $(y \to x) \vee (x \to y) = 1$ because $\mathcal{A}$ is regular. Thus $\mathcal{A}$ is a MTL-algebra. □

Some properties of the semi-preliearity (cont.)

There are examples showing that a semi-prelinear residuated lattice is not always a De Morgan residuated lattice, a pseudocomplemented residuated lattice nor a Stonean residuated lattice.

Some properties of the semi-preliearity (cont.)

There are examples showing that a semi-prelinear residuated lattice is not always a De Morgan residuated lattice, a pseudocomplemented residuated lattice nor a Stonean residuated lattice.

Proposition 6

Let $\mathcal{A}$ be a De Morgan and pseudocomplemented residuated lattice. Then $\mathcal{A}$ is a semi-prelinear residuated lattice.

Proof.

Let $x, y \in A$. We have by (P5),$(x' \to y') \vee (y' \to x') = (x' \odot y)' \vee (y' \odot x)'$. Since $\mathcal{A}$ is a De Morgan residuated lattice, then $(x' \to y') \vee (y' \to x') = [(x' \odot y) \wedge (y' \odot x)]'$. Moreover, by (P1), we have $x' \odot y \leq x' \wedge y$ and $y' \odot x \leq y' \wedge x$. Therefore $(x' \odot y) \wedge (y' \odot x) \leq (x' \wedge y) \wedge (y' \wedge x) = x' \wedge (y \wedge y') \wedge x = 0$, because $\mathcal{A}$ is a pseudocomplemented residuated lattice. Which implies that, $(x' \to y') \vee (y' \to x') = 0' = 1$. Thus $\mathcal{A}$ is a semi-prelinear residuated lattice. □

Summarizing the interrelation between some subclasses of residuated lattice

- Every MTL-algebra is a semi-prelinear residuated lattice.
- Every semi-prelinear residuated lattice and regular residuated lattice is a MTL-algebra.
- A De Morgan residuated lattice is not always semi-prelinear.
- A semi-prelinear residuated lattice is not always a De Morgan residuated lattice, a pseudocomplemented residuated lattice nor a Stonean residuated lattice.
- A De Morgan and pseudocomplemented residuated lattice is a semi-prelinear residuated lattice.
- Every Stonean residuated lattice is a semi-prelinear residuated lattice. But the converse is not always true.

Other properties of the semi-prelinearity

Recall that a proper ideal P of a residuated lattice $\mathcal{A}$ is called :

- a **prime ideal** of $\mathcal{A}$ if P is a prime element of $(\mathcal{I}(\mathcal{A}), \subseteq)$, that is, if I, J are ideals of $\mathcal{A}$ and $I \cap J \subseteq P$, then $I \subseteq P$ or $J \subseteq P$.
- a **prime ideal of the second kind** of $\mathcal{A}$ if for every $x, y \in A$, $x \wedge y \in P$ implies $x \in I$ or $y \in P$.
- is a **prime ideal of third kind** of $\mathcal{A}$ if for all $x, y \in A$, $(x \to y)' \in P$ or $(y \to x)' \in P$.

Other properties of the semi-prelinearity

Recall that a proper ideal P of a residuated lattice $\mathcal{A}$ is called :

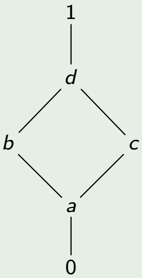
- a **prime ideal** of $\mathcal{A}$ if P is a prime element of $(\mathcal{I}(\mathcal{A}), \subseteq)$, that is, if I, J are ideals of $\mathcal{A}$ and $I \cap J \subseteq P$, then $I \subseteq P$ or $J \subseteq P$.
- a **prime ideal of the second kind** of $\mathcal{A}$ if for every $x, y \in A$, $x \wedge y \in P$ implies $x \in I$ or $y \in P$.
- is a **prime ideal of third kind** of $\mathcal{A}$ if for all $x, y \in A$, $(x \to y)' \in P$ or $(y \to x)' \in P$.

Contrary to the situation in MTL-algebras, in semi-prelinear residuated lattices, the notion of prime ideal and prime ideal of second kind are not equivalent.

Example 6

Let $A = \{0, a, b, c, d, 1\}$ be a set with the lattice structure presented in figure to the right, the operations $\rightarrow$ and $\odot$ defined in tables.

- $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a semi-prelinear residuated lattice.
- We have $(b \rightarrow c) \vee (c \rightarrow b) = d \vee b = d \neq 1$ then $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is not a MTL-algebra.
- The ideal $I = \{0\}$ is a prime ideal which is not a prime ideal of the second kind, because $(b \rightarrow c)' = d' = b \notin I$ and $(c \rightarrow b)' = b' = d \notin I$; and since $(b \rightarrow c)' \wedge (c \rightarrow b)' = b \vee d = b \notin I$, it is neither a prime ideal of the third kind.



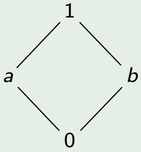
$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	d	d	1	d	1	1
c	b	b	b	1	1	1
d	b	b	b	d	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	0	b
c	0	0	0	c	c	c
d	0	0	0	c	c	d
1	0	a	b	c	d	1

Example 6

Let $A = \{0, a, b, 1\}$ be a set with the lattice structure presented in figure to the right, the operations $\rightarrow$ and $\odot$ defined in tables.

- $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a semi-prelinear residuated lattice.
- The ideal $I = \{0\}$ is a prime ideal of the third kind which is not a prime ideal of the second kind, because $(b \rightarrow a)' = a' = b \notin I$ and $(a \rightarrow b)' = b' = a \notin I$; nor a prime ideal.
- The filter $F = \{1\}$ is a prime filter of the third kind which is not a prime filter of the second kind, because $b \rightarrow a = a \notin F$ and $a \rightarrow b = b \notin F$; nor a prime filter.



$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Table of contents

Numbers of residuated lattices with selected properties

	1	2	3	4	5	6	7	8	9	10	11	12
all RLs	1	1	2	7	26	129	723	4712	34698	290565	2779183	30653419
S-MTL	1	1	2	7	25	122	676	4355	31661	261231	2458286	26680775
S-IDE	1	1	1	3	8	30	143	794	5090	37036	306456	2897889
S-DIV	1	1	2	6	19	80	394	2261	14721	108022	893895	8402176
DMO	1	1	2	7	23	110	567	3370	22131	160838	1294365	11656964
STO	1	1	1	3	7	27	129	726	4713	34705	290565	2779212

Residuated Lattices and Residuated Multilattices with Applications

Pr. Célestin Lélé, CIMPA-ICTP course

June 17-20, 2024

CIMPA

Course content

Chapter 1 : Lattices
Chapter 2 : Residuated lattices
Chapter 3 : Some new classes on residuated lattices
Chapter 4 : Multilattices
Chapter 5 : Residuated multilattices
Chapter 6 : Subclasses of residuated multilattices
Chapter 7 : Applications

Chapter 4 : Multilattices

Table of contents

Definition 1

A **lattice** is a partial ordered set in which every pair of elements a and b has a least upper bound $a \vee b$ and a greatest lower bound $a \wedge b$.

Let $(M, \leq)$ be a poset and $X \subseteq M$, we will be denoted $U(X)$ the set of upper bounds and $L(X)$ the set of lower bounds of X in $(M, \leq)$.

Ordered multilattices

- A **multi-supremum** of X is a minimal element of the set of upper bounds of X and a **multi-infimum** is a maximal element of the set of lower bounds of X,
- The set of multi-suprema of X is denoted by **multisup(X)** and the set of multi-infima of X is denoted by **multiinf(X)**.

Ordered multilattices

- A **multi-supremum** of X is a minimal element of the set of upper bounds of X and a **multi-infimum** is a maximal element of the set of lower bounds of X,
- The set of multi-suprema of X is denoted by **multisup(X)** and the set of multi-infima of X is denoted by **multiinf(X)**.

Definition 2

A poset $(M, \leq)$ is an **ordered multilattice** if and only if it satisfies that:

(1) for all $a, b, c \in M$, $a \leq c$ and $b \leq c$ implies that there exists $x \in multisup\{a, b\}$ such that $x \leq c$.

(2) for all $a, b, c \in M$, $c \leq a$ and $c \leq b$ implies that there exists $x \in multiinf\{a, b\}$ such that $c \leq x$.

Algebraic multilattices

We will write $a \sqcup b$ to denote $Multisup\{a, b\}$ and $a \sqcap b$ to denote $Multiinf\{a, b\}$ and we usually write $a \sqcup b = x$ instead of $a \sqcup b = \{x\}$ and $a \sqcap b = y$ instead of $a \sqcap b = \{y\}$ when $a \sqcup b$ or $a \sqcap b$ is a singleton

Definition 3

.Let M be a nonempty set. We call binary non-deterministic operator(hence nd-operator) on M any application $F : M \times M \to 2^M$.

Algebraic multilattices

Similarly to the lattice theory, it is possible to give an algebraic version of multilattice as $(M; \sqcup, \sqcap)$, where M is a nonempty set and $\sqcap, \sqcup$ satisfy suitable conditions.

Algebraic multilattices

Definition 4

An **algebraic join-multisemilattice** $(M, \sqcup)$ is a set M with binary nd-operator $\sqcup$ on M. That satisfying the following properties:

(1) **Idempotency:** $a \sqcup a = \{a\}$, for all $a \in M$,

(2) **Commutativity:** $a \sqcup b = b \sqcup a$, for all $a, b \in M$,

(3) **Left m-associativity:** if: $a \sqcup b = \{b\}$, then $(a \sqcup b) \sqcup c \subseteq a \sqcup (b \sqcup c)$ for all $a, b, c \in M$,
Right m-associativity: if: $b \sqcup c = \{c\}$, then $a \sqcup (b \sqcup c) \subseteq (a \sqcup b) \sqcup c$ for all $a, b, c \in M$,
m-associativity: if it is Left and Right m-associative.

(4) **Comparability:**

a) if $c \in a \sqcup b$, then $a \sqcup c = b \sqcup c = \{c\}$,

b) If $c_1, c_2 \in x \sqcup y$ and $c_1 \sqcup c_2 = \{c_1\}$, then $c_1 = c_2$.

Algebraic multilattices

Definition 5

Let M be a nonempty set, and let $\sqcap$, $\sqcup$ be two nd-operators on M. If for all $a, b \in M$, $a \sqcap (a \sqcup b) = a$ and $a \sqcup (a \sqcap b) = a$, then $(\sqcap, \sqcup)$ satisfies the **absorption** laws.

An **algebraic meet-multisemilattice** is obtained by duality of the above properties. Now, we can define an **algebraic multilattice** as a poset that satisfies the absorption law and both definitions of join-multisemilattice and meet-multisemilattice.

Algebraic multilattices

Given algebraic multilattice $\mathcal{M} := (M; \sqcup, \sqcap)$, if we define the binary relation by:

$$a \leq b \text{ if and only if } a \sqcup b = \{b\},$$

we obtain the ordered version of multilattice. Both definitions of multilattices are proved to be equivalent.

The algebraic and order structures of a multilattice will be used simultaneously, even if only one notation is given.

Definition 6

- A multillatice is said to be **full** if $a \sqcup b \neq \emptyset$ and $a \sqcap b \neq \emptyset$ for all $a, b \in M$,
- A multilattice is **bounded** if it has a least element $\perp$ and a greatest element $\top$.

When $a \sqcup b$ (resp. $a \sqcap b$) is a singleton $\{x\}$ (resp. $\{y\}$), we write $a \sqcup b = x$ (resp. $a \sqcap b = y$).

Example 7

An example of bounded full multilattice which is not a lattice is the multilattice with the following Hasse diagram:

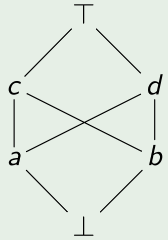


Figure 1: Hasse diagram of multilattice ML_6

Definition 8 (Martinez, 2005)

Let $\mathcal{M} := (M; \sqcap, \sqcup)$ be a multilattice and X be a nonempty subset of M.

1) X is called a **full sub-multilattice** (f-sub-multilattice) of M if for all $a, b \in X, \emptyset \neq a \sqcup b \subseteq X$ and $\emptyset \neq a \sqcap b \subseteq X$,
2) X is called a **restricted sub-multilattice** (r-sub-multilattice) of M if for all $a, b \in X, (a \sqcup b) \cap X \neq \emptyset$ and $(a \sqcap b) \cap X \neq \emptyset$.

It is obvious that every f-sub-multilattice is a r-sub-multilattice, but not the converse.

Table of contents

Definition of Ideal

Definition 9

Let $\mathcal{M} := (M; \sqcup, \sqcap)$ be a multilattice. A nonvoid subset I of M is called an **M-ideal**, if the following conditions hold:

1) For all $x, y \in I$, $x \sqcup y \subseteq I$,
2) For all $a \in I$ and for all $x \in M$, $x \sqcap a \subseteq I$,
3) For all $x, y \in M$, if $(x \sqcap y) \cap I \neq \emptyset$, then $x \sqcap y \subseteq I$.

Definition of filter

Definition 10

. Let $\mathcal{M} := (M; \sqcup, \sqcap)$ be a multilattice. A nonvoid subset F of M is called a **m-filter**, if the following conditions hold:

1) For all $x, y \in F$, $x \sqcap y \subseteq F$,
2) For all $a \in F$ and for all $x \in M$, $x \sqcup a \subseteq F$,
3) For all $x, y \in M$, if $(x \sqcup y) \cap F \neq \emptyset$, then $x \sqcap y \subseteq F$.

We refer to a multilattice filter as an **m-filter** and a multilattice ideal as an **m-ideal**.

Definition of congruence relation

Remark 11

- *Let M be a non empty set and $\equiv$ be a binary relation on M and $X, Y \subseteq M$, then $X \widehat{\equiv} Y$ denotes that, for all $x \in X$, there exists $y \in Y$ such that $x \equiv y$ and for all $y \in Y$ there exists $x \in X$ such that $x \equiv y$.*
- *If $\equiv$ is an equivalence relation on M, we write $[x]$ to denote the equivalence class of an element $x \in M$.*

Definition 12

Let $(M, \sqcap, \sqcup)$ be a multilattice. A congruence $\equiv$ on M is any equivalence relation such that if $x \equiv y$, then $x \sqcap z \widehat{\equiv} y \sqcap z$ and $x \sqcup z \widehat{\equiv} y \sqcup z$, for all $x, y, z \in M$.

Let $\equiv$ be a congruence relation defined on a multilattice $(M, \sqcap, \sqcup)$ and $M/\equiv$ be the set of equivalence classes. $M/\equiv$ has a multilattice structure, with the order relation defined by $[x] \preceq [y] :\Leftrightarrow \emptyset \neq x \sqcup y \subseteq [y] \Leftrightarrow x \sqcup y \widehat{\equiv} y$.

Proposition 13

Let $\equiv$ be a congruence relation on a multilattice $(M, \sqcap, \sqcup)$. Then for all $x, y \in M$, the following conditions are equivalent:

1. $[x] \preceq [y]$;
2. *for all* $a \in [x]$, *there exists* $b \in [y]$ *such that* $a \leq b$;
3. *for all* $b \in [y]$, *there exists* $a \in [x]$ *such that* $a \leq b$.

Definition of homomorphism

Definition 14

Let $h : M \to M^{'}$ be a map between the underlying sets of $\mathcal{M} := (M; \sqcup, \sqcap)$ and $\mathcal{M}^{'} := (M^{'}; \sqcup^{'}, \sqcap^{'})$.

1) h is a **join-homomorphism** if $h(a \sqcup b) \subseteq h(a) \sqcup^{'} h(b)$ for all $a, b \in M$,
2) h is a **meet-homomorphism** if $h(a \sqcap b) \subseteq h(a) \sqcap^{'} h(b)$ for all $a, b \in M$,
3) h is a **homomorphism** if and only if $h(a \sqcup b) \subseteq h(a) \sqcup^{'} h(b)$ and $h(a \sqcap b) \subseteq h(a) \sqcap^{'} h(b)$ for all $a, b \in M$.

Definition of homomorphism

Proposition 15

Let $h : M \to M'$ *be a map between the underlying sets of* $\mathcal{M} := (M; \sqcup, \sqcap)$ *and* $\mathcal{M}' := (M'; \sqcup', \sqcap')$, *where* $\mathcal{M}$ *is full. Then* h *is a homomorphism if and only if, for all* $a, b \in M$*:*

1) $h(a \sqcup b) = (h(a) \sqcup' h(b)) \cap h(M)$,
2) $h(a \sqcap b) = (h(a) \sqcap' h(b)) \cap h(M)$.

References

- M. Benado, *Les ensembles partiellement ordonnés et le théorème de raffinement de Schreier. I.*, Ĉehoslovack. mat. Ẑ, 79(1954), 105-129.
- I. P. Cabrera, P. Cordero, G. Gutierrez, J. Martinez, M. Ojeda-Aciego, *Congruence relations on multilattices*, Computational Intelligence in Decision and Control, (2008)139-144.
- I. J. Johnston, *Some results involving multilattice Ideal and distributivity*, Discrete Mathematics, 83(1990), 27-35.

Residuated Lattices and Residuated Multilattices with Applications

Pr. Célestin Lélé, CIMPA-ICTP course

June 17-20, 2024

CIMPA

Chapter 5 : Residuated multilattices

Table of contents

Introduction

A residuated multilattice is a partially ordered commutative residuated integral monoid (a.k.a pocrim) whose poset is a multilattice. In other words, residuated multilattices combine in a delicate manner the pocrim and multilattice structures on the same set. Therefore residuated multilattices generalize both residuated lattices and multilattices. Cabrera et al. laid the ground work on the topic by introducing the main properties and also studying filters within the new framework.

Table of contents

Definition 1

A **pocrim** *is a poset* $(A, \leq)$ *with a maximum element 1 and two binary operations* $\odot, \rightarrow$ *such that* $(A, \odot, 1)$ *is a commutative integral monoid, and for all* $a, b, c \in A$, $a \odot c \leq b$ *if and only if* $c \leq a \rightarrow b$.

It follows that $a \leq b$ if and only if $a \rightarrow b = 1$. If a pocrim A has a bottom element 0, then it is said to be bounded, and for all $x \in M$ and $X \subseteq M$, we note $x' := x \rightarrow 0$ and $X' = \{x' \mid x \in X\}$.

Generalities

Theorem 2

For any bounded pocrim $\mathcal{A} = (A; \leq, \odot, \rightarrow, 0, 1)$*, the following properties hold for every* $x, y, z \in A$ *:*

(P1) $x \odot y \leq x, y$ *and* $y \leq x \rightarrow y$;

(P2) $x \odot (x \rightarrow y) \leq x, y$ *and* $x, y \leq (x \rightarrow y) \rightarrow y$;

(P3) $x \leq y$ *implies* $x \odot z \leq y \odot z$, $z \rightarrow x \leq z \rightarrow y$, $y \rightarrow z \leq x \rightarrow z$;

(P4) *If* $x \leq y$ *then* $y' \leq x'$;

(P5) $(x \odot y)' = x \rightarrow y' = y \rightarrow x' = x'' \rightarrow y'$;

(P6) $0' = 1, 1' = 0, x' = x''', x \leq x''$;

(P7) $x \odot x' = 0$ *and* $x \odot y = 0$ *iff* $x \leq y'$;

(P8) $x' \rightarrow y' = y'' \rightarrow x''$ *and* $x'' \leq x' \rightarrow y'$;

(P9) $x \rightarrow y \leq y' \rightarrow x'$;

(P10) $x \to x = 1$;

(P11) $x \to y \leq (z \to x) \to (z \to y)$ and $x \to y \leq (y \to z) \to (x \to z)$.

Definition 3

Given $\mathcal{A} = (A, \leq, \odot, \to, 0, 1)$ a pocrim. A non-empty subset $F \subseteq A$ is a **p-filter** *(pocrim filter) of $\mathcal{A}$ if the following conditions hold:*

(i) *if $a, b \in F$, then $a \odot b \in F$;*

(ii) *if $a \leq b$ and $a \in F$, then $b \in F$.*

F is a **deductive system** *if it satisfies :*

(1) *$1 \in F$;*

(2) *$a \to b \in F$ and $a \in F$, imply $b \in F$, for all $a, b \in A$.*

In a pocrim, the notions of p-filter and deductive system are equivalent.

Definition 4

A **residuated multilattice** $\mathcal{M} := (M, \leq, \odot, \rightarrow, 1)$ *is a pocrim whose underlying poset is a multilattice. A residuated multilattice is* **bounded** *if it has a minimum element 0. A residuated multilattice is called pure if its underlying multilattice is pure.*

It is clear that every residuated multilattice is a full multilattice. For convenience and to increase the readability, we summarize the main properties of residuated multilattices needed throughout.

Proposition 5

In a bounded residuated multilattice $\mathcal{M}$, the following conditions hold, for all $x, y, z \in M$.

(M1) $x \odot y; x \odot (x \to y) \in (x \sqcap y)\downarrow;$

(M2) $(x \odot y) \sqcup (x \odot z) \subseteq x \odot (y \sqcup z);$

(M3) $(x \sqcap y) \to z \subseteq [(x \to z) \sqcup (y \to z)]\uparrow;$

(M4) $(x \sqcup y) \to z \subseteq [(x \to z) \sqcap (y \to z)]\downarrow;$

(M5) $(x \to y) \sqcap (y \to z) \subseteq (x \sqcup y) \to z;$

(M6) $x \to y = max\{(x \sqcup y) \to y\} = max\{x \to (x \sqcap y)\};$

(M7) $(x \sqcap y)' \subseteq (x' \sqcup y')\uparrow;$

(M8) $(x \sqcup y)' \subseteq (x' \sqcap y')\downarrow;$

(M9) $x' \sqcap y' \subseteq (x \sqcup y)'.$

Definition 6

Let $\mathcal{M}$ be a residuated multilattice and X a non empty subset of M.
X is called a **full sub-residuated multilattice** (or **f-Sub-**$\mathcal{RML}$) if the following conditions hold:

(FS1) $1 \in X$

(FS2) For all $x, y \in X$, $x \odot y \in X$ and $x \to y \in X$

(FS3) For all $x, y \in X$, $x \sqcup y \subseteq X$ and $x \sqcap y \subseteq X$.

X is called a **restricted sub-residuated multilattice** (or **r-Sub-**$\mathcal{RML}$) if the following conditions hold:

(RS1) $1 \in X$

(RS2) For all $x, y \in X$, $x \odot y \in X$ and $x \to y \in X$

(RS3) For all $x, y \in X$, $(x \sqcup y) \cap X \neq \emptyset$ and $(x \sqcap y) \cap X \neq \emptyset$.

Proposition 7

Let $\mathcal{M}$ be a full multilattice. Then $\mathcal{M}$ can be seen as a lattice if and only if $\mathcal{M}$ can be seen as a $\wedge$-semilattice (i.e., $a \sqcap b$ is a singleton for all $a, b \in M$) if and only if $\mathcal{M}$ can be seen as a $\vee$-semilattice (i.e., $a \sqcup b$ is a singleton for all $a, b \in M$). In particular a residuated multilattice can be seen as a residuated lattice if and only if $\mathcal{M}$ can be seen as a $\wedge$-semilattice if and only if $\mathcal{M}$ can be seen as a $\vee$-semilattice.

Definition 8

Let $\mathcal{M} = (M, \sqcap, \sqcup, \odot, \rightarrow, 1)$ be a residuated multilattice. A congruence relation $\equiv$ on a residuated multilattice $\mathcal{M}$ is a congruence relation on a multilattice $(M, \sqcap, \sqcup)$ such that for all $x, y \in M$, if $x \equiv y$, then $x \odot z \equiv y \odot z$, $x \rightarrow z \equiv y \rightarrow z$ and $z \rightarrow x \equiv z \rightarrow y$.

Proposition 9

Let $\mathcal{M} = (M, \sqcap, \sqcup, \odot, \rightarrow, 1)$ *be a residuated multilattice and* $\equiv$ *be a congruence relation on* $\mathcal{M}$*. Then* $(M/\equiv; \preceq, \odot, \rightarrow)$ *is a residuated multilattice with the operations* $[x] \sqcap [y] = \{[a], a \in x \sqcap y\}$, $[x] \sqcup [y] = \{[a], a \in x \sqcup y\}$, $[x] \rightarrow [y] = [x \rightarrow y]$ *and* $[x] \odot [y] = [x \odot y]$.

Definition 10

Let $\mathcal{M} = (M, \sqcap, \sqcup, \odot, \rightarrow, 1)$ be a residuated multilattice. A non-empty set $F \subseteq M$ is said to be **filter** if it is a deductive system and the following condition holds : $x \rightarrow y \in F$ implies $(x \sqcup y) \rightarrow y \subseteq F$ and $x \rightarrow (x \sqcap y) \subseteq F$.

Generalities

Theorem 11

Let M be a residuated multilattice and F a deductive system of M, F is a filter if and only if :

1. *F is a m-filter;*
   and for all $a, b \in M$,
2. *for all $x, y \in a \sqcup b$, if $x \to y \in F$, then $y \to x \in F$;*
3. *for all $x, y \in a \sqcap b$, if $x \to y \in F$, then $y \to x \in F$.*

Definition 12

Let $\mathcal{M}$ be a residuated multitreillis. A filter F of $\mathcal{M}$ is said to be **consistent** *if the following conditions hold for all $a, b, c \in M$:*

(1) *if $a \to c$, $b \to c \in F$, then $(a \sqcup b) \to c \subseteq F$*

(2) *if $c \to a$, $c \to b \in F$, then $c \to (a \sqcap b) \subseteq F$*

Theorem 13

Let $\mathcal{M}$ be a residuated multilattice and F be a filter of $\mathcal{M}$. Then, the relation $x \equiv_F y$ if and only if $a \to b, b \to a \in F$ defines a congruence relation of the residuated multilattice $\mathcal{M}$.

Definition 14

. Let $\mathcal{M} := (M; \leq, \odot_M, \to_M, 1_M)$ and $\mathcal{M}' := (M'; \leq^{'} \odot_{M'}, \to_{M'}, 1_{M'})$ be two residuated multilattices. A map $h : M \to M^{'}$ is said to be a **homomorphism** between $\mathcal{M}$ and $\mathcal{M}'$ if it meets the following conditions:

- h is a multilattice homomorphism,
- for all $a, b \in M$, $h(a \to b) = h(a) \to h(b), h(a \odot b) = h(a) \odot h(b)$.

Remark 15

For all homomorphisms $h : M \to M^{'}$ *between residuated multilattices* $\mathcal{M} := (M; \leq, \odot, \to, 1)$ *and* $\mathcal{M}' := (M'; \leq, \odot, \to, 1')$*, For all homomorphism between residuated multilattices* $\mathcal{M}$ *and* $\mathcal{M}^{'}$*, one can observe that* $h(1) = 1'$*.*

Table of contents

Ordinal sums of Residuated Multilattices

The goal of this section is to extend the ordinal sum construction of pocrims to residuated multilattices and obtain a general procedure to construct new residuated multilattices.

Small residuated multilattice

Proposition 16

There is no residuated multilattice structure on ML_6 extending its existing partial order.

Proposition 16

There is no residuated multilattice structure on ML_6 extending its existing partial order.

Proof. Suppose that there is a pocrim structure on ML_6. We first use the fact that in any residuated multilattice, $y \leq x \to y$, $x \to (y \to z) = y \to (x \to z)$ and $x \to y = max\{x \to (x \sqcap y)\}$.

Small residuated multilattice (cont.)

(i) Since $c \leq d \to c \neq \top$, then $d \to c = c$. Similarly $c \to d = d$.

(ii) Note that since $b \leq c \to b$ and $b \leq d \to b$, then $c \to b \in \{b, c, d\}$ and $d \to b \in \{b, c, d\}$. Note that $c \to b \leq c \to d = d$, hence $c \to b \neq c$. Likewise, $d \to b \neq d$. Thus $c \to b \in \{b, d\}$ and $d \to b \in \{b, c\}$. We show that $c \to b = b$ and $d \to b = b$ by showing that the other combinations lead to contradiction.

- If $c \to b = b$ and $d \to b = c$, then $c = d \to b = d \to (c \to b) = c \to (d \to b) = c \to c = \top$, which is a contradiction.
- If $c \to b = d$ and $d \to b = b$, then $\top = d \to d = d \to (c \to b) = c \to (d \to b) = c \to b$, which implies that $c \leq b$. This is impossible.
- If $c \to b = d$ and $d \to b = c$, then $\top = a \to c = a \to (d \to b) = d \to (a \to b)$. So $d \leq a \to b \neq \top$, hence $d = a \to b$. Now, $\top = a \to d = a \to (c \to b) = c \to (a \to b) = c \to d$, which is again a contradiction.

  Since a and b play symmetrical roles, we deduce $c \to a = a$ and $d \to a = a$.

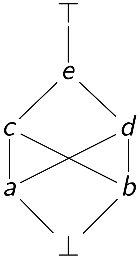
(iii) We have $c \to d = max\{c \to (c \sqcap d)\} = max\{c \to a, c \to b\} = max\{a, b\}$, which is not possible because a and b are not comparable.

Small residuated multilattice (cont.)

By Proposition 16 we obtain that every bounded pure $\mathcal{RML}$ has order greater than or equal to 7. The next example shows that there is at least one bounded pure $\mathcal{RML}$ of order 7, which shall subsequently be denoted by RML_7.

Example 17

Let ML_7 be the multilattice depicted in the following figure:



Ordinal sums of Residuated Multilattices

The operations $\odot$ and $\rightarrow$ are defined as follows:

$\odot$	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	a
b	$\bot$	$\bot$	b	b	b	b	b
c	$\bot$	$\bot$	b	b	b	b	c
d	$\bot$	$\bot$	b	b	b	b	d
e	$\bot$	$\bot$	b	b	b	b	e
$\top$	$\bot$	a	b	c	d	e	$\top$

$\rightarrow$	$\bot$	a	b	c	d	e	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
a	e	$\top$	e	$\top$	$\top$	$\top$	$\top$
b	a	a	$\top$	$\top$	$\top$	$\top$	$\top$
c	a	a	e	$\top$	e	$\top$	$\top$
d	a	a	e	e	$\top$	$\top$	$\top$
e	a	a	e	e	e	$\top$	$\top$
$\top$	$\bot$	a	b	c	d	e	$\top$

Then it is readily verified that RML_7 is an $\mathcal{RML}$.

Next, we apply the ordinal sum construction to $\mathcal{RML}$s.

Ordinal sums of Residuated Multilattices

First, we recall the construction of the ordinal sum of pocrims. Let $(A, \leq_A, \odot_A, \rightarrow_A, \bot_A, \top_A)$ and $(B, \leq_B, \odot_B, \rightarrow_B, \bot_B, \top_B)$ be two bounded pocrims. Consider the set $C := (A \amalg B)/\{\top_A \equiv \bot_B\}$ (that is the disjoint union of A and B with $\top_A$ and $\bot_B$ identified) and define $\leq$ on C by $x \leq y$ if $x, y \in A$ and $x \leq_A y$, or $x, y \in B$ and $x \leq_B y$ or $x \in A \setminus \{\top_A\}$ and $y \in B$. Define $\odot$ and $\rightarrow$ on C by:

$$x \odot y = \begin{cases} x \odot_A y & \text{if } x, y \in A \\ x \odot_B y & \text{if } x, y \in B \\ x & \text{if } x \in A \setminus \{\top_A\} \textit{ and } y \in B \\ y & \text{if } y \in A \setminus \{\top_A\} \textit{ and } x \in B \end{cases}$$

$$x \rightarrow y = \begin{cases} x \rightarrow_A y & \text{if } x, y \in A \\ x \rightarrow_B y & \text{if } x, y \in B \\ \top_B & \text{if } x \in A \setminus \{\top_A\} \textit{ and } y \in B \\ y & \text{if } y \in A \setminus \{\top_A\} \textit{ and } x \in B \end{cases}$$

Ordinal sums of Residuated Multilattices (cont.)

Then $(C, \leq, \odot, \rightarrow, \bot_A, \top_B)$ is a pocrim, and it is called the ordinal sum of A and B and denoted by $A \oplus B$. Our goal is to apply this construction to create new $\mathcal{RML}$s from old ones. The first step in this process consists in identifying the smallest pure $\mathcal{RML}$.

Ordinal sums of Residuated Multilattices (cont.)

Proposition 18

Let M and N be bounded $\mathcal{RML}$s. Then

(1) *The ordinal sum $M \oplus N$ (as pocrims) is an $\mathcal{RML}$.*

(2) *M is an f-Sub-$\mathcal{RML}$ of $M \oplus N$ if and only if M has a unique coatom.*

(3) *N is a filter of $M \oplus N$, in particular N is an f-Sub-$\mathcal{RML}$.*

(4) *The quotient $\mathcal{RML}$ $M \oplus N/N$ is canonically isomorphic to M.*

(5) *N is a consistent filter of $M \oplus N$ if and only if $\mathcal{M}$ is a residuated lattice.*

Proof

Let M, N be two $\mathcal{RML}$s.

Ordinal sums of Residuated Multilattices

(1) We know that the ordinal sum of $\mathcal{M}$ and $\mathcal{N}$ (as pocrims) is again a pocrim. In addition, it is clear that with respect to the ordinal sum's order $\sqcup_{M\oplus N}$ and $\sqcap_{M\oplus N}$ are given by:

$$x \sqcup_{M\oplus N} y = \begin{cases} x \sqcup_M y & \text{if } x, y \in M \text{ and } x \sqcup_M y \neq \top_M \\ x \sqcup_N y & \text{if } x, y \in N \\ y & \text{if } x \in M \setminus \{\top_M\}, \quad y \in N \\ \bot_{\mathcal{N}} & \text{if } x, y \in M \setminus \{\top_M\} \text{ and } x \sqcup y = \top_M \end{cases}$$

$$x \sqcap_{M\oplus N} y = \begin{cases} x \sqcap_M y & \text{if } x, y \in M \\ x \sqcap_N y & \text{if } x, y \in N \\ x & \text{if } x \in M, \quad y \in N \end{cases}$$

It remains to show that for every $x, y, a, b \in M \oplus N$, with $x, y \leq a$ and $b \leq x, y$, there exists $u \in x \sqcup_{M\oplus N} y$ and $v \in x \sqcap_{M\oplus N} y$ such that $u \leq a$ and $b \leq v$. This is easily verified by considering the cases $a \in M \setminus \{\top_M\}$, $a \in N$ and $x, y \in M \setminus \{\top_M\}$ or $x, y \in N$ or $x \in M \setminus \{\top_M\}$ and $y \in N$ as necessary. Therefore $M \oplus N$ is a $\mathcal{RML}$ as claimed.

Ordinal sums of Residuated Multilattices

(2) Assume there exists $x_0 \neq y_0$ coatoms in M. Then $x_0 \sqcup_{M \oplus N} y_0 = \bot_{\mathcal{N}} \notin M$. Thus M is not a f-Sub-$\mathcal{RML}$ of $M \oplus N$.
Conversely suppose that M has a unique coatom a. Then for all $x, y \in M \setminus \{\top_M\}$, $x \sqcup_{M \oplus N} y = x \sqcup_M y \subseteq \downarrow a \subseteq M$. If $x = \top_M$ or $y = \top_M$, then $x \sqcup_{M \oplus N} y = \top_M \in M$. It is clear from the description of $\sqcap_{M \oplus N}$ above that $x \sqcap_{M \oplus N} y \subseteq M$ for all $x, y \in M$. In addition it follows from the definition of the ordinal sum of pocrims that $x \to y, x \odot y \in M$ for all $x, y \in M$. Therefore, M is an f-Sub-$\mathcal{RML}$ of $M \oplus N$.

(3) To show that N is a filter of $M \oplus N$, we first show that N is a deductive system. It follows from the definition of $\odot$ in the ordinal sum that N is $\odot$-closed, and from the definition of the order that whenever $x \leq y$ with $x \in N$, then $y \in N$. So, N is a deductive system. Moreover, let $x, y \in M \oplus N$ such that $x \to y \in N$. By definition of $\to$ in the ordinal sum, either $x, y \in N$ and $x \to y = x \to_N y$ or $x \to y = \top_N$. If $x \to y = x \to_N y$ with $x, y \in N$ then from the description of $\sqcup_{M \oplus N}$ above and that of $\to$ in the ordinal sum, it follows that $x \sqcup_{M \oplus N} y \to y \subseteq N$ and $x \to x \sqcap_{M \oplus N} y \subseteq N$. If $x \to y = \top_N$, then $x \leq y$. Thus, $x \sqcup y \to y = y \to y = \top_N \in N$ and $x \to x \sqcap y = x \to x = \top_N$.

Ordinal sums of Residuated Multilattices

Thus N is a filter of $M \oplus N$.

(4) From the definition of $\rightarrow$ is the ordinal sum, it is clear that for every $x \in M \oplus N$, $[x]_N = \{x\}$ if $x \in M \setminus \{\top_M\}$ and $[x]_N = N$ otherwise. Now consider $f : M \oplus N/N \rightarrow M$ defined by:

$$f([x]_N) = \begin{cases} x & \text{if } \; x \in M \setminus \{\top_M\} \\ \top_M & \quad \text{otherwise} \end{cases}$$

An elementary but lengthy argument shows that f is a well-defined isomorphism of $\mathcal{RML}$s.

(5) We recall that a filter F of an $\mathcal{RML}$ $\mathcal{M}$ is consistent if and only if M/F is a residuated lattice. Therefore, the result is a consequence of (4).

Table of contents

References

- I. P. Cabrera, P. Cordero, G. Gutierrez, J. Martinez, M. Ojeda-Aciego, *On residuation in multilattices: Filters, congruences, and homomorphisms*, Fuzzy Sets Systems, 234 (2014)1-21.
- B. B. Koguep Njionou, L. Kwuida, C. Lele, *Formal Concepts and Residuation on Multilattices*, Fundamenta Informaticae 188(2022), 4, 1–21.
- L. N. Maffeu Nzoda *residuated multilattices and aplications to Formal Concept Analysis*, PhD dissertation, university of Dschang (2018).

Residuated Lattices and Residuated Multilattices with Applications

Pr. Célestin Lélé, CIMPA-ICTP course

June 17-20, 2024

CIMPA

Chapter 6 : Subclasses of residuated multilattices

Table of contents

Introduction

As a generalization of residuated lattices, it seems natural to consider various measures of the gap between the two types of systems. One such measure can be formulated in terms of which additional properties of residuated multilattices would force the structure down to residuated lattices. Note that one instance of such consideration was investigated in by Cabrera et al. when the authors showed that a residuated multilattice with idempotent product (i. e., $x \odot x = x$, for all x) is a Heyting algebra. We seek to expand this approach and explore the effect of adding an equation on a residuated multilattice. We shall discover that in some cases, the equation forces the structure to collapse down to the corresponding class of residuated lattice as in the case of the above-mentioned reference. We also obtain in other cases, new classes of residuated multilattices containing examples of pure residuated multilattices and properly containing the corresponding well-known classes of residuated lattices. We hope this work extends and deepens the foundations laid by Cabrera et al and sets the stage for more in-depth studies of residuated multilattices.

Table of contents

Semi-idempotent residuated multilattice

Cabrera et al. showed that if a residuated multilattice has an idempotent multiplication, then the structure collapses down to a residuated lattice. This suggests that what would have been a natural generalization of idempotent to residuated multilattices is too strong, and a more appropriate concept to study in this context seems to be that of semi-idempotent residuated multilattices.

Definition 1

Let $\mathcal{M}$ be a residuated multilattice. $\mathcal{M}$ is semi-idempotent if for all $x \in M$, $(x' \odot x')' = x''$.

Example 2

Let $M = \{a_i, b_j, 0 \leq i \leq 3 \text{ and } 0 \leq j \leq 6\}$ be the multilattice depicted in the figure 1.

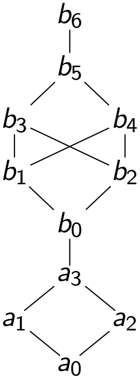


Figure 1: ML_{11}

Semi-idempotent residuated multilattice

Consider the sets $A = \{a_i, 0 \leq i \leq 3\}$; $B = \{b_j, 0 \leq j \leq 6\}$ and $C = \{b_j, 2 \leq j \leq 5\}$. The operations $\odot$ and $\rightarrow$ are defined as follows:

$$x \odot y = \begin{cases} x \wedge y, & \textit{if} \quad x, y \in A \\ b_0, & \textit{if} \quad (x \in \{b_0, b_1\} \text{ and } y \in B \setminus \{b_6\}) \text{ or } (y \in \{b_0, b_1\} \text{ and } x \in B \setminus \{b_6\}) \\ x, & \textit{if} \quad y = b_6 \text{ or } (x \in A \text{ and } y \in B \setminus \{b_6\}) \\ y, & \textit{if} \quad x = b_6 \text{ or } (y \in A \text{ and } x \in B \setminus \{b_6\}) \\ b_2, & \quad \textit{otherwise} \end{cases}$$

and

$$x \rightarrow y = \begin{cases} b_6, & \textit{if} \quad x \leq y \\ y, & \textit{if} \quad x = b_6 \text{ or } (x = a_3 \text{ and } y \in \{a_0, a_1, a_2\}) \text{ or } (x \in B \setminus \{b_6\} \text{ and } y \in A) \\ a_1, & \textit{if} \quad x = a_2 \text{ and } y \in \{a_0, a_1\} \\ a_2, & \textit{if} \quad x = a_1 \text{ and } y \in \{a_0, a_2\} \\ b_1, & \textit{if} \quad x \in C \text{ and } y \in \{b_0, b_1\} \\ b_5, & \quad \textit{otherwise} \end{cases}$$

Semi-idempotent residuated multilattice

We claim that when equipped with $\odot$ and $\rightarrow$, $\mathcal{M}$ is a semi-idempotent residuated multilattice. Indeed, one has $(a_0' \odot a_0')' = (b_6 \odot b_6)' = b_6' = a_0''$;
$(a_1' \odot a_1')' = (a_2 \odot a_2)' = (a_2 \wedge a_2)' = a_2' = a_1''$; $(a_2' \odot a_2')' = (a_1 \odot a_1)' = (a_1 \wedge a_1)' = a_1' = a_2''$;
for all $x \in \{a_3, b_j, 0 \leq j \leq 6\}$, $(x' \odot x')' = (a_0 \odot a_0)' = a_0' = x''$.

Girard-monoid residuated lattices are also known in the literature under the name regular or involutive residuated lattices. We wish to extend this notion to residuated multilattices.

Definition 3

A residuated multilattice $\mathcal{M}$ is called a Girard-monoid residuated multilattice if it satisfies $x'' = x$, for all $x \in M$.

The next example shows a pure Girard-monoid residuated mutilattice.

Semi-idempotent residuated multilattice

Example 4

Let $M = (\{0\} \times \mathbb{Z}^+) \cup (\mathbb{Z}^+ \times \mathbb{Z}) \cup (\{1\} \times \mathbb{Z}^-)$ be a multilattice, where the partial order on M is defined by

$$\langle \alpha, i \rangle \leq \langle \beta, j \rangle \text{ iff } i + |\alpha - \beta| \leq j$$

Then M becomes a residuated multilattice under the operations $\odot$ and $\to$ as defined below.

$$\begin{array}{rcll} x \odot y & = & y \odot x & \\ \langle 1, i \rangle \odot \langle 1, j \rangle & = & \langle 1, i + j \rangle & (i, j \leq 0) \\ \langle 1, i \rangle \odot \langle \alpha, j \rangle & = & \langle \alpha, i + j \rangle & (i \leq 0) \\ \langle 1, i \rangle \odot \langle 0, j \rangle & = & \langle 0, \max\{0, i + j\} \rangle & (i \leq 0 \leq j) \\ \langle \alpha, i \rangle \odot \langle \beta, j \rangle & = & \langle 0, \max\{0, i + j + |\alpha - \beta|\} \rangle & \\ \langle \alpha, i \rangle \odot \langle 0, j \rangle & = & \langle 0, k \rangle \odot \langle 0, j \rangle = \langle 1, 0 \rangle & (0 \leq j, k) \end{array}$$

Semi-idempotent residuated multilattice

$$
\begin{array}{rcll}
x \to y & = & \langle 1,0 \rangle & \text{iff } \; x \leq y \\
\langle 1,i \rangle \to \langle 1,j \rangle & = & \langle 1, \min\{0, j-i\} \rangle & (i,j \leq 0) \\
\langle 1,i \rangle \to \langle \alpha,j \rangle & = & \langle \alpha, j-i \rangle & (i \leq 0) \\
\langle 1,i \rangle \to \langle 0,j \rangle & = & \langle 0, j-i \rangle & (i \leq 0 \leq j) \\
\langle \alpha,i \rangle \to \langle \beta,j \rangle & = & \langle 1, \min\{0, j-i-|\alpha-\beta|\} \rangle & \\
\langle \alpha,i \rangle \to \langle 0,j \rangle & = & \langle \alpha, j-i \rangle & (0 \leq j) \\
\langle 0,i \rangle \to \langle 0,j \rangle & = & \langle 1, \min\{0, j-i\} \rangle & (0 \leq i,j)
\end{array}
$$

We claim that for all $x \in M, x'' = x$.

(i) If $x = \langle 1, i\rangle$ we have $x'' = \langle 1, i\rangle'' = (\langle 1, i\rangle \to \langle 0, 0\rangle) \to \langle 0, 0\rangle = \langle 0, -i\rangle \to \langle 0, 0\rangle = \langle 1, i\rangle$

(ii) If $x = \langle \alpha, i\rangle$ we have $x'' = \langle \alpha, i\rangle'' = (\langle \alpha, i\rangle \to \langle 0, 0\rangle) \to \langle 0, 0\rangle = \langle \alpha, -i\rangle \to \langle 0, 0\rangle = \langle \alpha, i\rangle$

(iii) If $x = \langle 0, i\rangle$ we have $x'' = \langle 0, i\rangle'' = (\langle 0, i\rangle \to \langle 0, 0\rangle) \to \langle 0, 0\rangle = \langle 1, -i\rangle \to \langle 0, 0\rangle = \langle 0, i\rangle$

Semi-idempotent residuated multilattice

Therefore, $\mathcal{M}$ is a pure Girard-monoid residuated multilattice as claimed.

We examine next the overlap of the introduced two classes.

Proposition 5

Semi-idempotent and Girard-monoid residuated mutilattices are regular and idempotent residuated lattices.

Proof

Let $\mathcal{M}$ be a Girard-monoid residuated mutilattice that is a semi-idempotent. Let $x \in M$, as $\mathcal{M}$ is a semi-idempotent, we have $((x')' \odot (x')')' = (x')''$; i.e., $(x'' \odot x'')' = x''' = x'$ and $(x'' \odot x'')'' = x''$. Since $\mathcal{M}$ is a Girard-monoid residuated mutilattice, then $x \odot x = x$, and $\mathcal{M}$ is an idempotent. It follows that $\mathcal{M}$ is a residuated lattice, indeed a Heyting algebra. Thus, $\mathcal{M}$ becomes a regular and idempotent residuated lattice.

Semi-idempotent residuated multilattice

Proposition 6

In a residuated multilattice $\mathcal{M}$, the following properties are equivalent:

(i) $x' \to x'' = x''$, *for all* $x \in M$.

(ii) $(x' \to y') \to x'' = y' \to x'' \in (x \sqcup y)''$, *for all* $x, y \in M$.

Continuing in the same logic, we introduce pseudocomplemented residuated multilattices.

Definition 7

A residuated multilattice $\mathcal{M}$ is pseudocomplemented if for all $x \in M$, $x \sqcap x' = \{0\}$.

Among other things, we show that semi-idempotent residuated multillatices and pseudocomplemented residuated multillatices are the same class.

Semi-idempotent residuated multilattice

Proposition 8

Let $\mathcal{M}$ be a residuated multilattice. The following assertions are equivalent:

(i) *$\mathcal{M}$ is a semi-idempotent;*

(ii) *For every $x \in M$, $(x \odot x)' = x'$;*

(iii) *For every $x, y \in M$, $(x' \to y') \to x'' = y' \to x'' \in (x \sqcup y)''$;*

(iv) *For every $x, y \in M$, $(x \odot y)' \in (x \sqcap y)'$;*

(v) *$\mathcal{M}$ is pseudocomplemented.*

Proof

Observe that (ii) $\to$ (i) is clear. In addition, $(i) \Leftrightarrow (iii)$ follows from Proposition 6 and property (P5).

Semi-idempotent residuated multilattice

We have $x \odot x \leq x$ then $x' \leq (x \odot x)'$. Moreover, we have $(x \odot x)' = x \to x' \leq x'' \to x' = (x')' \to (x')''$. By hypothesis, we have $(x')' \to (x')'' \in (x' \sqcup x')'' = \{(x')''\} = \{x'\}$, then $(x \odot x)' \leq x'$. Therefore, $(x \odot x)' = x'$.

(ii) $\Rightarrow$ (iv) Suppose that for all $x \in M$, $(x \odot x)' = x'$. Let $x, y \in M$ and $t \in x \sqcap y$. We have $t \leq x, y$, then $t \odot t \leq x \odot y$; therefore $(x \odot y)' \leq (t \odot t)' = t'$, according to the hypothesis. Moreover, $x \odot y \in (x \sqcap y)\downarrow$, then there exists $t_0 \in x \sqcap y$ such that $x \odot y \leq t_0$, then $t_0' \leq (x \odot y)'$. Therefore, $(x \odot y)' = t_0' \in (x \sqcap y)'$.

(iv) $\Rightarrow$ (v) Suppose that for all $x, y \in M$, $(x \odot y)' \in (x \sqcap y)'$. Let $x \in M$. By hypothesis, we have $(x \odot x')' \in (x \sqcap x')'$, by (P7) and (P6), we obtain $1 \in (x \sqcap x')'$, therefore there exists $t \in x \sqcap x'$ such that $t' = 1$; i.e., there exists $t \in x \sqcap x'$ such that $t = 0$. Thus, since 0 is a bottom element and by definition of $\sqcap$, we obtain $x \sqcap x' = \{0\}$, hence $\mathcal{M}$ is pseudocomplemented.

(v) $\Rightarrow$ (ii) Suppose that $\mathcal{M}$ is pseudocomplemented. Let $x \in M$. We have $\{x \to x'\} = (x \to x) \sqcap (x \to x') \subseteq x \to (x \sqcap x')$. By hypothesis, $x \sqcap x' = \{0\}$, then $\{x \to x'\} \subseteq \{x \to 0\}$, i.e., $(x \odot x)' = x \to x' = x'$.

Table of contents

Semi-divisible residuated multilattices

As emphasized in the previous chapters, if a residuated multilattice has an idempotent multiplication, the structure collapses down to a residuated lattice, and more precisely to a Heyting algebra. This motivates the consideration of other classes of residuated multilattices that collapse to subclasses of residuated lattices under the addition of a single property. We start with the following property that is motivated by divisibility in residuated lattices.

$$x \odot (x \to y) \in x \sqcap y \qquad (D)$$

One of our goals for this section is to show that if a residuated multilattice satisfies (D), then it becomes a divisible residuated lattice.

Semi-divisible residuated multilattices

Proposition 9

Let $\mathcal{M}$ be a residuated multilattice. Then $\mathcal{M}$ satisfies (D) if and only if for all $x, y \in M$ such that $x \leq y$, there exists $z \in M$ such that $x = y \odot z$.

Proof

$\Rightarrow$) Assume that $\mathcal{M}$ satisfies (D) and let $x, y \in M$ such that $x \leq y$. Then by the condition (D), we have $y \odot (y \to x) \in x \sqcap y = \{x\}$. Therefore, $x = y \odot (y \to x)$.
$\Leftarrow$) Assume that for all $x, y \in M$ such that $x \leq y$, there exists $z \in M$ such that $x = y \odot z$. Given $a, b \in M$, we have $a \odot (a \to b) \leq a, b$. Then there exists $c \in a \sqcap b$ such that $a \odot (a \to b) \leq c$. On the other hand, since $c \leq a$, by hypothesis $c = a \odot d$ for some $d \in M$. Hence, $a \odot d \leq b$ and $d \leq a \to b$. Thus, $c = a \odot d \leq a \odot (a \to b)$ and $c = a \odot (a \to b)$.

Semi-divisible residuated multilattices

Proposition 10

Residuated multilattices satisfying (D) coincide with divisible residuated lattices.

Proof

Suppose that $\mathcal{M}$ is a residuated multilattice satisfying (D). Let $a, b \in M$, then $a \odot (a \to b) \leq a, b$. Hence, there exists $x \in a \sqcap b$ such that $a \odot (a \to b) \leq x$. Now, let $y \in a \sqcap b$, then $y \leq a$ and by Proposition 9, $y = a \odot t$ for some $t \in M$. In particular, $a \odot t \leq b$ which implies that $t \leq a \to b$. Hence, $y \leq a \odot (a \to b)$. Therefore, $a \odot (a \to b)$ is the maximum element of $a \sqcap b$, which means that $a \sqcap b$ is a singleton and $a \sqcap b = a \odot (a \to b)$. Given $x \in a \sqcup b$ then $a, b \leq x$. By Proposition 9 and $(M2)$ there exist $y, z \in M$ such that $a = x \odot y$ and $b = x \odot z$. We obtain $a \sqcup b = (x \odot y) \sqcup (x \odot z) \subseteq x \odot (y \sqcup z)$. If $a \sqcup b$ is a singleton there is nothing to prove. Otherwise, let $x' \in a \sqcup b$ such that $x \neq x'$.

Semi-divisible residuated multilattices

As $x' \in a \sqcup b \subseteq x \odot (y \sqcup z)$ there exists $t \in y \sqcup z$ such that $x' = x \odot t$. Therefore, $x' \leq x$ which implies $x' = x$.

Since the notion of divisibility is essentially lost in the setting of residuated multilattices, we introduce the following weaker notion that appears to be more adequate.

Definition 11

A residuated multilattice $\mathcal{M}$ is semi-divisible if it satisfies (SD) for all $x, y \in M$.

$$[x' \odot (x' \to y')]' \in (x' \sqcap y')' \quad (SD)$$

The next example shows that unlike residuated multilattices satisfying the property (D), there exists pure residuated multilattices that are semi-divisible.

Example 12

Let us consider the multilattice ML_7 depicted in the figure below:

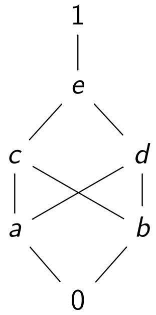


Figure 2: ML_7

Semi-divisible residuated multilattices

If we define the operations $\odot$ and $\to$ as follows:

$$x \odot y = \begin{cases} 0 & \text{if} \quad x, y \in \{0, a, b, c, d, e\} \\ x & \text{if} \quad y = 1 \\ y & \text{if} \quad x = 1 \end{cases} \qquad \text{and} \quad x \to y = \begin{cases} 1 & \text{if} \quad x \leq y \\ y & \text{if} \quad x = 1 \\ e & \text{if} \quad \text{otherwise} \end{cases}$$

One can verify that under the operations $\odot$ and $\to$, ML_7 is a semi-divisible residuated multilattice.

It turns out that the class of semi-divisible residuated multilattices contains all semi-idempotent residuated multilattices as we show next.

Semi-divisible residuated multilattices

Proposition 13

Every semi-idempotent residuated multilattice is a semi-divisible residuated multilattice.

Proof

Let $\mathcal{M}$ be a semi-idempotent residuated multilattice. Let $x, y \in M$. Since, $x' \odot (x' \to y') \in (x' \sqcap y')\downarrow$, there exists $t \in x' \sqcap y'$ such that $x' \odot (x' \to y') \leq t$, which implies that $t' \leq [x' \odot (x' \to y')]'$. Let us show that $[x' \odot (x' \to y')]' \leq t'$. Note that $(t \odot t)' = t'$, as $\mathcal{M}$ is semi-idempotent. Then,
$[x' \odot (x' \to y')]' \to t' = [x' \odot (x' \to y')]' \to (t \odot t)' \geq (t \odot t) \to x' \odot (x' \to y')$ by (M13).
Since $t \leq x', y'$, then $t \odot t \leq x' \odot y'$, therefore
$x' \odot y' \to x' \odot (x' \to y') \leq t \odot t \to x' \odot (x' \to y')$ by (M12). Moreover, by (M12) and (M14), $1 = y' \to (x' \to y') \leq x' \odot y' \to x' \odot (x' \to y')$. Then, $1 = [x' \odot (x' \to y')]' \to t'$, i.e., $[x' \odot (x' \to y')]' \leq t'$. Thus, $[x' \odot (x' \to y')]' = t' \in (x' \sqcap y')'$ and $\mathcal{M}$ is a semi-divisible residuated multilattice.

Semi-divisible residuated multilattices

Remark 14

Note that the converse of the above proposition is not always true. In fact, the semi-divisible residuated multilattice of Example 12 is not a semi-idempotent because we have $(a' \odot a')' = 1 \neq e = a''$.

Proposition 15

Let $\mathcal{M}$ *be a residuated multilattice such that* $(x \to y) \to y \in x \sqcup y$*, for all* $x, y \in M$*. Then,* $\mathcal{M}$ *is semi-divisible.*

Proof

Assume that for all $x, y \in M, (x \to y) \to y \in x \sqcup y$. For all $x, y \in M, x' \odot (x' \to y') \leq x', y'$. Therefore, there exists $z \in x' \sqcap y'$ such that $x' \odot (x' \to y') \leq z$, which implies that $z' \leq [x' \odot (x' \to y')]'$.

Semi-divisible residuated multilattices

In addition,
$[x' \odot (x' \to y')]' = (x' \to y') \to (x' \to 0) = (x' \to y') \to x'' = (y'' \to x'') \to x'' \in x'' \sqcup y''$.
Now, by $(M8)$, $z' \in (x' \sqcap y')' \subseteq [(x'' \sqcup y'')]\!\uparrow$ and there exists $t \in x'' \sqcup y''$ such that $t \leq z' \leq [x' \odot (x' \to y')]'$. Since, $t, [x' \odot (x' \to y')]' \in x'' \sqcup y''$ we obtain $t = z' = [x' \odot (x' \to y')]'$. Thus, $[x' \odot (x' \to y')]' \in x'' \sqcup y''$.

Remark 16

In the proof above, it was established that the equation $[x' \odot (x' \to y')]' = (y'' \to x'') \to x''$ *holds in any residuated multilattice. Thus a residuated multilattice* $\mathcal{M}$ *is semi-divisible if and only if* $(y'' \to x'') \to x'' \in (x' \sqcap y')'$ *for all* $x, y \in M$.

Table of contents

Prelinear residuated multilattices

Unlike idempotency and divisibility, the prelinearity condition does not collapse in residuated multilattices setting.

Definition 17

A residuated multilattice $\mathcal{M}$ *is prelinear if it satisfies the prelinear identity : for all* $x, y \in M$, $(x \to y) \sqcup (y \to x) = \{1\}$.

The next example displays a pure prelinear residuated multilattice with 12 elements.

Example 18

Consider the mutillatice depicted in the Hasse diagram together with the operations $\odot$ *and* $\to$.

Prelinear residuated multilattices

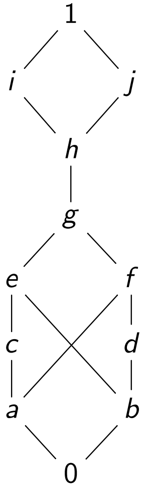


Figure 3: ML_{12}

$\odot$	0	a	b	c	d	e	f	g	h	i	j	1
0	0	0	0	0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	0	0	0	a	a
b	0	0	0	0	0	0	0	0	0	b	0	b
c	0	0	0	0	0	0	0	0	0	0	c	c
d	0	0	0	0	0	0	0	0	0	d	0	d
e	0	0	0	0	0	0	0	0	0	b	c	e
f	0	0	0	0	0	0	0	0	0	d	a	f
g	0	0	0	0	0	0	0	0	0	d	c	g
h	0	0	0	0	0	0	0	0	h	h	h	h
i	0	0	b	0	d	b	d	d	h	i	h	i
j	0	a	0	c	0	c	a	c	h	h	j	j
1	0	a	b	c	d	e	f	g	h	i	j	1

Prelinear residuated multilattices

$\to$	0	a	b	c	d	e	f	g	h	i	j	1
0	1	1	1	1	1	1	1	1	1	1	1	1
a	i	1	i	1	i	1	1	1	1	1	1	1
b	j	j	1	j	1	1	1	1	1	1	1	1
c	i	i	i	1	i	1	i	1	1	1	1	1
d	j	j	j	j	1	j	1	1	1	1	1	1
e	h	h	i	j	i	1	i	1	1	1	1	1
f	h	j	h	j	i	j	1	1	1	1	1	1
g	h	h	h	j	i	j	i	1	1	1	1	1
h	g	g	g	g	g	g	g	g	1	1	1	1
i	c	c	e	c	g	e	g	g	j	1	j	1
j	d	f	d	g	d	g	f	g	i	i	1	1
1	0	a	b	c	d	e	f	g	h	i	j	1

It is readily verified that this yields a prelinear residuated multilattice.

Prelinear residuated multilattices

In the next series of results, we explore some properties of prelinear residuated multilattices.

Proposition 19

Let $\mathcal{M}$ be a prelinear residuated multilattice. Then, for all $x, y, z \in M$,

$$(z \odot (x \to y)) \sqcup (z \odot (y \to x)) = ((x \to y) \to z) \sqcap ((y \to x) \to z) = \{z\}$$

Proof

Let $x, y, z \in M$. We have, by (M2)
$(z \odot (x \to y)) \sqcup (z \odot (y \to x)) \subseteq z \odot ((x \to y) \sqcup (y \to x)) = \{z \odot 1\} = \{z\}$, as $\mathcal{M}$ is prelinear. That is, $(z \odot (x \to y)) \sqcup (z \odot (y \to x)) = \{z\}$. Moreover, we have, by (M5), $((x \to y) \to z) \sqcap ((y \to x) \to z) \subseteq ((x \to y) \sqcup (y \to x)) \to z = \{1 \to z\} = \{z\}$, because $\mathcal{M}$ is prelinear. Thus $((x \to y) \to z) \sqcap ((y \to x) \to z) = \{z\}$.

Prelinear residuated multilattices

Proposition 20

Let $\mathcal{M}$ be a prelinear residuated multilattice. Then, the following assertions are equivalent for all $x, y \in M$,

(i) $x \sqcup y$ *is a singleton;*

(ii) $((x \to y) \to y) \sqcap ((y \to x) \to x)$ *is a singleton;*

(iii) $x \sqcup y = ((x \to y) \to y) \sqcap ((y \to x) \to x)$*;*

(iv) $(x \sqcup y) \cap (((x \to y) \to y) \sqcap ((y \to x) \to x)) \neq \emptyset$.

Prelinear residuated multilattices

Proposition 21

Let $\mathcal{M}$ be a prelinear residuated multilattice. Then the following assertions are equivalent for all $x, y \in M$,

(i) $x \sqcap y$ *is a singleton;*

(ii) $(x \odot (x \to y)) \sqcup (y \odot (y \to x))$ *is a singleton;*

(iii) $x \sqcap y = (x \odot (x \to y)) \sqcup (y \odot (y \to x))$*;*

(iv) $(x \sqcap y) \cap ((x \odot (x \to y)) \sqcup (y \odot (y \to x))) \neq \emptyset$.

Table of contents

Semi-prelinear residuated multilattices

Definition 22

A residuated multilattice $\mathcal{M}$ is semi-prelinear if it satisfies the semi-prelinear identity: for all $x, y \in M$, $x, y \in M$, $(x' \to y') \sqcup (y' \to x') = \{1\}$

The next example shows that there exists pure residuated multilattices that are semi-prelinear but are not prelinear.

Example 23

Consider the mutillatice depicted in the Hasse diagram together with the operations $\odot$ *and* $\to$.

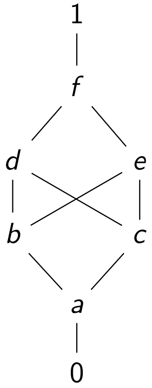


Figure 4: ML_8

Semi-prelinear residuated multilattices

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	a	a	a	a	a	a
b	0	a	a	a	a	a	a	b
c	0	a	a	a	a	a	a	c
d	0	a	a	a	a	a	a	d
e	0	a	a	a	a	a	a	e
f	0	a	a	a	a	a	a	f
1	0	a	b	c	d	e	f	1

$\rightarrow$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	0	1	1	1	1	1	1	1
b	0	f	1	f	1	1	1	1
c	0	f	f	1	1	1	1	1
d	0	f	f	f	1	f	1	1
e	0	f	f	f	f	1	1	1
f	0	f	f	f	f	f	1	1
1	0	a	b	c	d	e	f	1

It is readily verified that this yields a semi-prelinear residuated multilattice.

Semi-prelinear residuated multilattices

Proposition 24

Let $\mathcal{M}$ be a residuated multilattice. If $M' = \{x'; x \in M\}$ est linear, then $\mathcal{M}$ is semi-prelinear.

Proof

Suppose that $M' = \{x'; x \in M\}$ is linear. Then for all $x, y \in M$, $x' \leq y'$ or $y' \leq x'$, that is $x' \to y' = 1$ or $y' \to x' = 1$, which implies $(x' \to y') \sqcup (y' \to x') = \{1\}$. Therefore $\mathcal{M}$ is semi-prelinear.

Proposition 25

Every prelinear residuated multilattice is a semi-prelinear residuated multilattice

Semi-prelinear residuated multilattices

Proof

Let $\mathcal{M}$ be a prelinear residuated multilattice. Let $x, y \in M$, we have $x', y' \in M$, then $y') \sqcup (y' \to x') = \{1\}$, by hypothesis. Thus $\mathcal{M}$ is a semi-prelinear residuated multilattice.

Proposition 26

Every semi-prelinear and regular residuated multilattice is a prelinear residuated multilattice.

Proof

Let $\mathcal{M}$ be a residuated multilattice. Suppose that $\mathcal{M}$ is semi-prelinear and regular. For all $x, y \in M$, we have $(x' \to y') \sqcup (y' \to x') = \{1\}$. By P(8), $(y'' \to x'') \sqcup (x'' \to y'') = \{1\}$. Therefore $(x \to y) \sqcup (y \to x) = \{1\}$, because $\mathcal{M}$ is regular. Thus $\mathcal{M}$ is a prelinear residuated multilattice.

Table of contents

Stonean residuated multilattices

Definition 27

A residuated multilattice $\mathcal{M}$ is Stonean if for all $x \in M$, $x' \sqcup x'' = \{1\}$.

Example 28

The residuated multilattice ML_8 as defined in Example 23 is a Stonean RML.

Proposition 29

Let $\mathcal{M}$ be a Stonean residuated multilattice. Then for all $x \in M$, $x \sqcap x' = \{0\}$.

Proof

Let $x \in M$, we have by (M8), $(x' \sqcap x'')' \subseteq (x'' \sqcup x')\uparrow = \{1\}\uparrow = \{1\}$ because $\mathcal{M}$ is Stonean residuated multilattice. Therefore, $(x' \sqcap x'')' = \{1\}$, i.e., $x' \sqcap x'' = \{0\}$. Then $L(x', x'') = \{0\})$. Moreover, we have $L(x, x') \subseteq L(x', x'')$, thus $L(x, x') = \{0\}$, which implies that $x \sqcap y = \{0\}$.

Stonean residuated multilattices

Remark 30

By Proposition 29, any Stonean residuated multilattice is pseudocomplemented but the converse is not always true. We have the residuated multilattice of Example 2 which is a semi-idempotent and by Proposition 8, it is also a pseudocomplemented. However, this example is not a stonean residuated multilattice because $a_1'' \sqcup a_1' = a_1 \sqcup a_2 = \{a_3\} \neq \{b_6\}$.

Proposition 31

Let $\mathcal{M}$ *be a residuated multilattice. The following assertions are equivalent:*

(i) $\mathcal{M}$ *is Stonean;*

(ii) *for all* $x, y \in M$, $x' \sqcup y' = \{(x \odot y)'\}$;

(iii) *for all* $x, y \in M$, $(x \odot y)' \in x' \sqcup y'$.

Proof

(i) $\Rightarrow$ (ii) Suppose that $\mathcal{M}$ is Stonean and let $x, y \in M$. We have $x \odot y \leq x, y$, then $x', y' \leq (x \odot y)'$. Let $z \in M$ such that $x', y' \leq z$. We show that $(x \odot y)' \leq z$. We have $y'' = y' \to 0 \leq (x'' \to y') \to (x'' \to 0) = (x \odot y)' \to x'$. Since $x' \leq z$, then $(x \odot y)' \to x' \leq (x \odot y)' \to z$, therefore $y'' \leq (x \odot y)' \to z$. Moreover, $y' \leq z \leq (x \odot y)' \to z$, i.e., $(x \odot y)' \to z \in U(y', y'')$. By hypothesis, $y' \sqcup y'' = \{1\}$, i.e., $U(y', y'') = \{1\}$. Therefore, $(x \odot y)' \to z = 1$, which signifies that $(x \odot y)' \leq z$, then $x' \sqcup y' = \{(x \odot y)'\}$.
(ii) $\Rightarrow$ (iii) is trivial.
(iii) $\Rightarrow$ (i) Suppose that for all $x, y \in M$, $(x \odot y)' \in x' \sqcup y'$. Let $x \in M$, we have by hypothesis $(x \odot x')' \in x' \sqcup x''$, then $1 \in x' \sqcup x''$, because $(x \odot x')' = 0' = 1$. Therefore, by definition of $\sqcup$ and since 1 is a top element, we have $x' \sqcup x'' = \{1\}$. Thus $\mathcal{M}$ is Stonean.

Table of contents

Prime filters of residuated multilattices

Definition 32

Let $\mathcal{M}$ be a residuated multilatttice and F be a filter of $\mathcal{M}$. F is said to be a :

(i) *prime filter of the first kind if for all $x, y \in M$, $(x \sqcup y) \cap F \neq \emptyset$ implies $x \in F$ or $y \in F$.*

(ii) *prime filter of the second kind, if for all $x, y \in M$, $x \to y \in F$ or $y \to x \in F$.*

(iii) *prime filter of the third kind, if for all $x, y \in M$, $[(x \to y) \sqcup (y \to x)] \cap F \neq \emptyset$.*

Remark 33

By a previous result, any filter is a m-filter then by third item of the definition of m-filter, we conclude that a filter F of a residuated multilattice $\mathcal{M}$ is a :

- *prime filter of the first kind if and only if for all $x, y \in M$, $x \sqcup y \subseteq F$ implies $x \in F$ or $y \in F$.*
- *prime filter of the third kind, if and only if for all $x, y \in M$, $(x \to y) \sqcup (y \to x) \subseteq F$.*

Prime filters of residuated multilattices

Proposition 34

Let $\mathcal{M}$ be a residuated multilattice. Every prime filter of the second kind of $\mathcal{M}$ is also a prime filter of the first kind and a prime filter of the third kind.

Proof

Let F be a prime filter of the second kind of $\mathcal{M}$.
Let $x, y \in M$ such that $x \sqcup y \subseteq F$. By hypothesis, we have $x \to y \in F$ or $y \to x \in F$.
Suppose that $x \to y \in F$, then by the definition of filter, $x \sqcup y \to y \subseteq F$, i.e., for all $z \in x \sqcup y$, $z \to y \in F$. Since F is also a deductive system and $z \in F$ for all $z \in x \sqcup y$, we obtain $y \in F$.
Moreover, if $y \to x \in F$, by the same process, we show that $x \in F$. Therefore, $x \in F$ or $y \in F$. Thus, F is a prime filter of the first kind.
Let $x, y \in M$ and $t \in (x \to y) \sqcup (y \to x)$. We have $x \to y \leq t$ and $y \to x \leq t$. By hypothesis, $x \to y \in F$ or $y \to x \in F$, then $t \in F$ because F is also a p-filter. Thus, F is a prime filter of the third kind.

Prime filters of residuated multilattices

Proposition 35

Let $\mathcal{M}$ be a residuated multilattice and F be a filter of $\mathcal{M}$. Then, F is a prime filter of the second kind of $\mathcal{M}$ if and only if it is a prime filter of the first kind and a prime filter of the third kind.

Proof

$\Rightarrow$ This follows from Proposition 34.
$\Leftarrow$ Suppose that F is a prime filter of the first and third kinds. Let $x, y \in M$, as F is prime of third kind, we have $(x \rightarrow y) \sqcup (y \rightarrow x) \subseteq F$. Hence, since F is a prime filter of the first kind, we obtain $x \rightarrow y \in F$ or $y \rightarrow x \in F$.

Example 36

Let $M = \{0, a, b, c, d, e, f, g, h, 1\}$ *be the multilattice depicted in the following figure:*

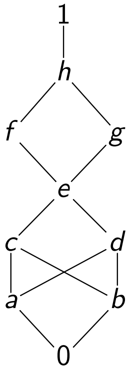


Figure 5: ML_{10}

Prime filters of residuated multilattices

Table 1: Table of operations $\odot$ and $\to$ defined on M

$\odot$	0	a	b	c	d	e	f	g	h	1
0	0	0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	0	0	a
b	0	0	0	0	0	0	0	0	0	b
c	0	0	0	0	0	0	0	0	0	c
d	0	0	0	0	0	0	0	0	0	d
e	0	0	0	0	0	0	0	0	0	e
f	0	0	0	0	0	0	f	0	f	f
g	0	0	0	0	0	0	0	g	g	g
h	0	0	0	0	0	0	f	g	h	h
1	0	a	b	c	d	e	f	g	h	1

$\to$	0	a	b	c	d	e	f	g	h	1
0	1	1	1	1	1	1	1	1	1	1
a	h	1	h	1	1	1	1	1	1	1
b	h	h	1	1	1	1	1	1	1	1
c	h	h	h	1	h	1	1	1	1	1
d	h	h	h	h	1	1	1	1	1	1
e	h	h	h	h	h	1	1	1	1	1
f	g	g	g	g	g	g	1	g	1	1
g	f	f	f	f	f	f	f	1	1	1
h	e	e	e	e	e	e	f	g	1	1
1	0	a	b	c	d	e	f	g	h	1

Prime filters of residuated multilattices

The subsets $\{f, h, 1\}$ and $\{g, h, 1\}$ are prime filters of the first, second and third kinds. Moreover $\{h, 1\}$ is a prime filter of the third kind, but it is not a prime filter of the second kind nor a prime ideal of the first kind. Then, $\{1\}$ is a prime filter of the first kind, but it is not a prime filter of the second kind nor a prime filter of the third kind.

Remark 37

In the above example, we observe that, in residuated multilattice, a prime filter of the first kind is not always a prime filter of the second kind nor a prime filter of the third kind. Also, a prime filter of the third kind is not always a prime filter of the second kind nor a prime filter of the first kind. The next diagram displays a visual of the comparison between the different types of prime filters.

Prime filters of residuated multilattices

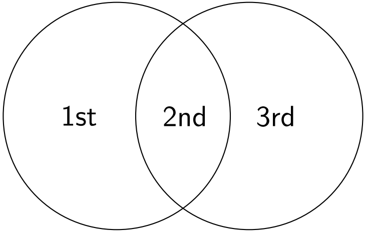


Figure 6: The relation between the different kinds of prime filters in residuated multilattices.

Prime filters of residuated multilattices

For prelinear residuated multilattices, the situation can be greatly simplified. Indeed, on one hand, it is clear that every filter of such a residuated multilattice is always prime of the third kind. On the other hand, prime filters of first and second kinds coincide.

Corollaire 38

Let $\mathcal{M}$ be a prelinear residuated multilattice and F be a filter of $\mathcal{M}$. Then, F is of the first kind if and only if it is a filter of the second kind.

Proof

This is an immediate consequence of Proposition 35 given that all filters of $\mathcal{M}$ are of the third kind.

Note that this situation generalizes what is known for residuated lattices.

Cabrera et al. have proved that the quotient of a residuated multilattice by a filter is a residuated lattice if and only if the filter is consistent. This motivates us to consider next the quotients of a residuated multilattices by the different kinds of prime filters.

Prime filters of residuated multilattices

Proposition 39

Let $\mathcal{M}$ be a residuated multilattice and F be a filter of $\mathcal{M}$. Then, $\mathcal{M}/\equiv_F$ is linear if and only if F is a prime filter of the second kind.

Proof

$\Rightarrow$) Suppose that $\mathcal{M}/\equiv_F$ is linear. Let $x, y \in M$, we have $[x] \preceq [y]$ or $[y] \preceq [x]$. If $[x] \preceq [y]$, then we have $x \sqcup y \subseteq [y]$, i.e., for all $z \in x \sqcup y$, $z \equiv_F y$. Let $z \in x \sqcup y$, we have $x \leq z$ and $z \to y \in F$, then $z \to y \leq x \to y \in F$. Therefore $x \to y \in F$. Else, if $[y] \preceq [x]$, we prove by the similar process that $y \to x \in F$. Thus F is a prime filter of the second kind.
$\Leftarrow$) Suppose that F is a prime filter of the second kind. Let $x, y \in M$ and $z \in x \sqcup y$. We have $x \to y \in F$ or $y \to x \in F$. If $x \to y \in F$, then by the definition of filter, $x \sqcup y \to y \subseteq F$, i.e., for all $z \in x \sqcup y$, $z \to y \in F$. Moreover, for all $z \in x \sqcup y$, $y \to z = 1 \in F$. Therefore, for all $z \in x \sqcup y$, $z \equiv_F y$, i.e., $x \sqcup y \widehat{\equiv} y$. Then $[x] \preceq [y]$. Else, if $y \to x \in F$, we show that $[y] \preceq [x]$. Thus $\mathcal{M}/\equiv_F$ is linear.

Remark 40

It follows from above proposition, that the quotient of a residuated multilattice by a filter collapses to a residuated lattice if and only if the filter is a prime filter of the second kind. Moreover, the latter quotient is linear. The following example illustrates that indeed for the other two kinds of prime filters, the quotient needs not be linear, nor collapse to a residuated lattice.

Example 41

Let us consider the residuated multilattice defined in Example 18. Let $F = \{h, 1\}$ *and* $G = \{1\}$. F *is a prime filter of third kind and* G *is a prime filter of first kind. We have* $\mathcal{M}/\equiv_F = \{\{0, a, b, c, d, e, f\}; \{f\}; \{g\}; \{h, 1\}\}$ *and* $\mathcal{M}/\equiv_G = \{\{0\}; \{a\}; \{b\}; \{c\}; \{d\}; \{e\}; \{f\}; \{g\}; \{h\}; \{1\}\}$. *Clearly, neither of the quotients is linear.*

Prime filters of residuated multilattices

Next, we show that pseudocomplemented, Stonean, semi-divisble and prelinear residuated multilattices are closed under the formation of quotients.

Proposition 42

Let $\mathcal{M}$ be a residuated multilattice and $\equiv$ be a congruence relation on $\mathcal{M}$.

(i). *If $\mathcal{M}$ is pseudocomplemented then $\mathcal{M}/\equiv$ is pseudocomplemented.*

(ii). *If $\mathcal{M}$ is Stonean then $\mathcal{M}/\equiv$ is Stonean.*

(iii). *If $\mathcal{M}$ is semi-divisible, then $\mathcal{M}/\equiv$ is semi-divisible;*

(iv). *If $\mathcal{M}$ is prelinear, then $\mathcal{M}/\equiv$ is prelinear.*

Table of contents

Computating the numbers of pure residuated multilattices

Table 2: Numbers of pure residuated multilattices

	1	2	3	4	5	6	7	8	9	10	11	12
all pure RMLs	0	0	0	0	0	0	6	128	2139	34987	604691	11512878
REG	0	0	0	0	0	0	0	3	4	78	149	1727
MTL	0	0	0	0	0	0	0	0	0	0	0	1
S-MTL	0	0	0	0	0	0	6	121	1957	30864	513255	9392552
S-DIV	0	0	0	0	0	0	2	50	833	12641	194576	3208695
S-IDE	0	0	0	0	0	0	0	6	131	2205	35996	618878
STO	0	0	0	0	0	0	0	6	128	2139	34987	1 604691

Table of contents

References

I. P. Cabrera, P. Cordero, G. Gutierrez, J. Martinez, M. Ojeda-Aciego, *Congruence relations on multilattices*, Computational Intelligence in Decision and Control, (2008)139-144.

B. B. Koguep Njionou, L. Kwuida, C. Lele, *Formal Concepts and Residuation on Multilattices*, Fundamenta Informaticae 188(2022), 4, 1–21

Residuated Lattices and Residuated Multilattices with Applications

Pr. Célestin Lélé, CIMPA-ICTP course

June 17-20, 2024

CIMPA

Course content

Chapter 7 : Applications

This chapter highlight application and utility of residuated multilattices in various domains.

This chapter highlight application and utility of residuated multilattices in various domains.

- Several generalizations of lattices have been investigated in the literature for their applications, most notably in the area of fuzzy logic programming, coding theory, data management and Formal Concept Analysis.

This chapter highlight application and utility of residuated multilattices in various domains.

- Several generalizations of lattices have been investigated in the literature for their applications, most notably in the area of fuzzy logic programming, coding theory, data management and Formal Concept Analysis.
- Various special residuated lattices are now used as the main structure of true values in fuzzy set theory and are subject to algebraic investigation; it is an important tool for the development of artificial intelligence, computer sciences and algebraic logic.

This chapter highlight application and utility of residuated multilattices in various domains.

- Several generalizations of lattices have been investigated in the literature for their applications, most notably in the area of fuzzy logic programming, coding theory, data management and Formal Concept Analysis.
- Various special residuated lattices are now used as the main structure of true values in fuzzy set theory and are subject to algebraic investigation; it is an important tool for the development of artificial intelligence, computer sciences and algebraic logic.
- One of the original motivations for the study of residuated lattices and residuated multilattices is their great importance in applications, particularly in fuzzy control, uncertain modeling, graph theory, coding theory , data visualisation and analysis.

This chapter highlight application and utility of residuated multilattices in various domains.

- Several generalizations of lattices have been investigated in the literature for their applications, most notably in the area of fuzzy logic programming, coding theory, data management and Formal Concept Analysis.
- Various special residuated lattices are now used as the main structure of true values in fuzzy set theory and are subject to algebraic investigation; it is an important tool for the development of artificial intelligence, computer sciences and algebraic logic.
- One of the original motivations for the study of residuated lattices and residuated multilattices is their great importance in applications, particularly in fuzzy control, uncertain modeling, graph theory, coding theory , data visualisation and analysis.
- More Clearly, lattices, residuated lattices, multilattices, residuated multilattices offer a rich mathematical framework that provide valuable tools for handling uncertainty and precision.

For each above area of application, we have the following summary:

[A] Formal Concept Analysis(FCA) Lattices, residuated lattices, multilattices, residuated multilattices helps to evaluate objets and attributes,They can also be used to build a residuated concept multilattice.

[1] Ganter B., and Wille,, R., Formal Concept Analysis: Mathematical Foundations Springer (1999)

[2] Belohlavek, R., Vychodil V., What is a fuzzy Concept Lattice?, Annals of Pure and Applied Logic 128(2004) 277 - 298

[3] Ghosh, P., Kundu, K., Sarkar, D. Fuzzy graph representation of a fuzzy concept lattice, Fuzzy Sets and Systems, 161(12), (2010), 1669-1675.

[4] Formica, A., Semantic Web Search based on rough sets and Fuzzy Formal Concept Analysis, Knowledge-Based system vol. 26, (2012), 40-47.

[5] Skopljanac-Macina, F., Blaskovic, B., Formal Concept Analyis - Overview and Applications, Procedia Engineering vol. 69, (2014), 1258 - 1267.

[6] Koguep, B.B.N., Kwuida, L., Lele, C.: Formal concepts and residuation on multilattices. Fundamenta Informaticae, vol. 188, (2023).

[B] Logic and kwowledge representation, lattices, residuated lattices, multilattices, residuated multilattices serve as truth-value sets.

[1] Bou, F., Esteva, F., Font, J. M., Gil, À. J., Godo, L., Torrens, A., Verdú, V. Logics preserving degrees of truth from varieties of residuated lattices. Journal of Logic and Computation, 19(6), (2009), 1031-1069.

[C] Pattern recognition, lattices, residuated lattices, multilattices, residuated multilattices help to identifying patterns in noisy data.

[D] Decision support systems, lattices, residuated lattices, multilattices, residuated multilattices serve for handhing imprecise information.

[1] Ignjatović, J.,Ćirić, M., Bogdanović, S., Determinization of fuzzy automata with membership values in complete residuated lattices, Information Sciences, 178(1), (2008), 164-180.

[2] Wu, L., Qiu, D., Automata theory based on complete residuated lattice-valued logic: Reduction and minimization, Fuzzy Sets and Systems, 161(12), (2010), 1635-1656.

[3] Wang, F., Feng, Z., Jiang, P., Reliable decentralized supervisory control of fuzzy discrete event systems, Fuzzy Sets and Systems, 161(12), (2010), 1657-1668.

[E] Fuzzy Logic: Residuated lattices, a generalization of which are residuated multilattices, play a critical role in the theory of fuzzy logic. They are used in logical algebras such as MTL-algebras, divisible residuated lattices, BL-algebras, MV-algebras, Heyting algebras, and Boolean algebras.

[F] Derivations on Residuated Multilattices: The study of derivations on residuated multilattices offer a different tool and perspective into the studies of residuated multilattices.