An Introduction to Quantum Groups and Hopf Algebras

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Lecture 1/4: What is a quantum group?

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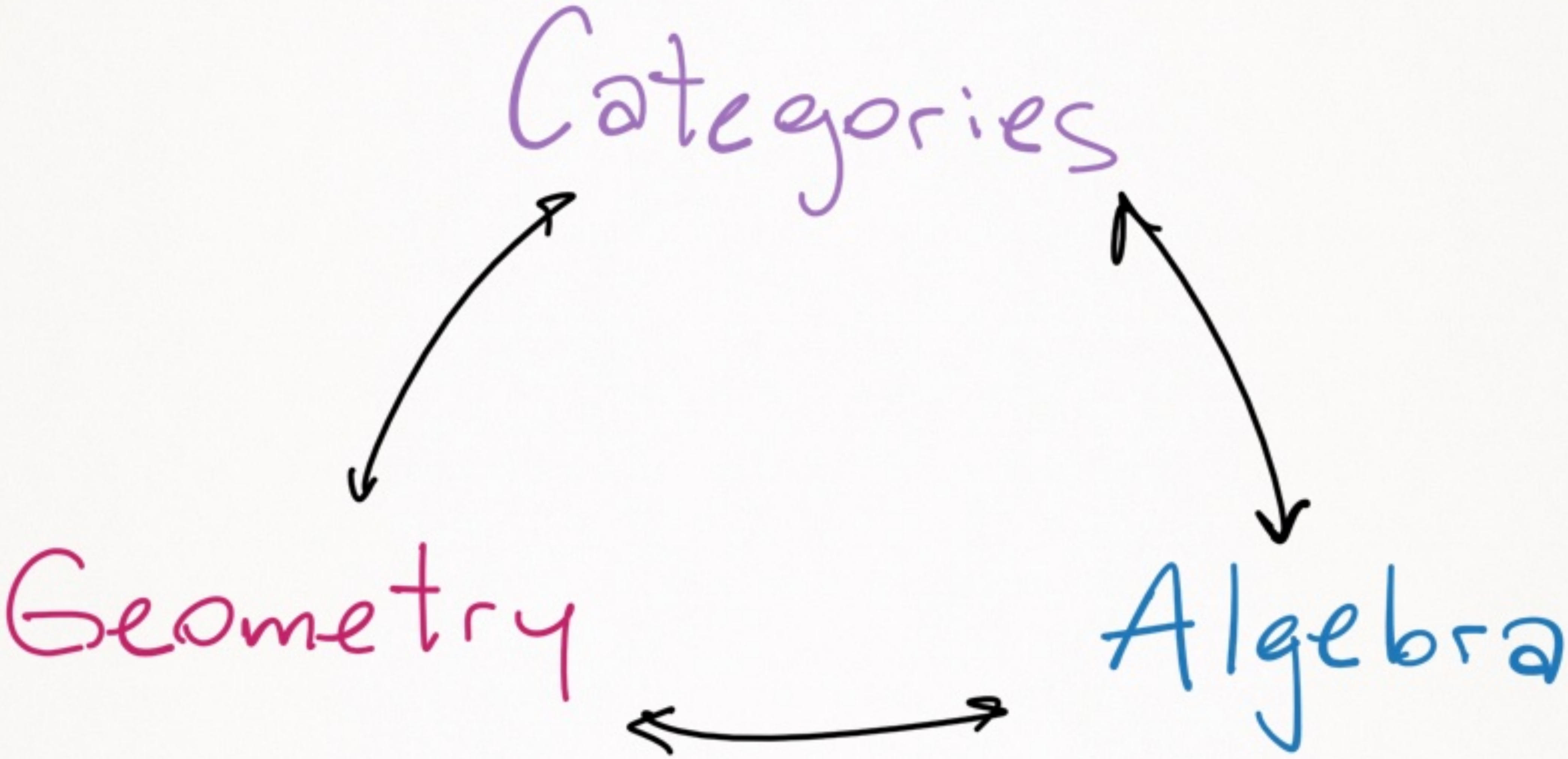
Dedicated to Pierre Cartier

(1932-2024)

(Photo Wikipedia)


Categories
Geometry
Algebra

The Classical Setting

Diccionary $\qquad$ $\Bbbk$ field $= \mathbb{R}, \mathbb{C}, \mathbb{F}_q$

Geometry $\longleftrightarrow$ Algebra

X set (finite, algebraic topological, etc) $\overset{\mathcal{O}}{\rightsquigarrow}$ $\mathcal{O}(X) = \{\alpha : X \to \Bbbk\}$

functions (rational, continuas, etc)

commutative algebra

$X \xrightarrow{f} Y$ $\rightsquigarrow$ $\mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$

function (algebraic, continuas differentiable, etc) $\qquad$ $\beta \longmapsto f^*(\beta)$

Diccionary $\Bbbk$ field $= \mathbb{R}, \mathbb{C}, \mathbb{F}_q$

Geometry $\longleftrightarrow$ Algebra

$$X = \mathrm{Spec}(A) \xleftarrow{\mathrm{Spec}(A)} A$$

A commutative $\Bbbk$-algebra

$= \{x : A \to \Bbbk \mid x \text{ algebra map}\}$

$= \mathrm{Alg}_{\Bbbk}(A, \Bbbk)$

Example: $M_n(\mathbb{k}) = n \times n$ matrices with coefficients in $\mathbb{k}$.

$$\mathcal{O}(M_n(\mathbb{k})) = \mathbb{k}[X_{ij} |\ 1 \leq i, j \leq n].$$

is the commutative ring of polynomials in n^2 variables.

$\mathcal{O}(M_n(\mathbb{k})) \subset \{\alpha : M_n(\mathbb{k}) \to \mathbb{k} \,|\, \alpha \text{ function}\}$ is a subalgebra.

X_{ij} is the function defined by the matrix coefficient

$$X_{ij}(A) = a_{ij} \qquad \forall\ A = (a_{ij})_{1 \leq i,j \leq n} \in M_n(\mathbb{k}).$$

Take the linear basis $\{E_{ij}\}_{1 \leq i,j \leq n}$ given by elementary matrix.
Then $\{X_{ij}\}_{1 \leq i,j \leq n}$ is the corresponding dual basis with

$$\langle X_{ij}, E_{kl} \rangle = \delta_{ik}\delta_{jl}.$$

$\mathcal{O}(M_n(\mathbb{k}))$ is the algebra of regular functions on $M_n(\mathbb{k})$

Consider $\mathrm{Spec}(\mathcal{O}(M_n(\mathbb{k})) = \mathrm{Alg}_{\mathbb{k}}(\mathcal{O}(M_n(\mathbb{k})), \mathbb{k})$.

Any algebra map $\chi : \mathcal{O}(M_n(\mathbb{k})) \to \mathbb{k}$ is defined on its values on the elements $\{X_{ij}\}_{1 \leq i,j \leq n}$.

This define a matrix $A_\chi = \big(\chi(X_{ij})\big)_{1 \leq i,j \leq n} \in M_n(\mathbb{k})$.

Conversely, any matrix $A = (a_{ij})_{1 \leq i,j \leq n} \in M_n(\mathbb{k})$ defines an algebra map $\chi_A : \mathcal{O}(M_n(\mathbb{k})) \to \mathbb{k}$ by $\chi_A(X_{ij}) = a_{ij}$ for all $1 \leq i, j \leq n$.

We have

$$\mathrm{Spec}(\mathcal{O}(M_n(\mathbb{k}))) = \mathrm{Alg}_{\mathbb{k}}(\mathcal{O}(M_n(\mathbb{k})), \mathbb{k}) = M_n(\mathbb{k})$$

Diccionary $\quad \mathbb{k}$ field $= \mathbb{R}, \mathbb{C}, \mathbb{F}_q$

Geometry $\longleftrightarrow$ Algebra

$X = G$ finite algebraic group $\qquad \mathcal{O}(G)$

- $G \times G \xrightarrow{m} G \qquad$ • $\mathcal{O}(G) \xrightarrow{m^*} \mathcal{O}(G) \otimes \mathcal{O}(G)$

$(gh)k = g(hk)$ assoc. $\qquad$ coassociative

- $\{e\} \xrightarrow{u} G$ unit $\qquad$ • $\mathcal{O}(G) \xrightarrow{u^*} \mathcal{O}(\{e\}) = \mathbb{k}$

$e \cdot g = g = g \cdot e \qquad$ counit

- $G \xrightarrow{(\ )^{-1}} G$ inverse $\qquad$ • $\mathcal{O}(G) \xrightarrow{S} \mathcal{O}(G)$

$g g^{-1} = e = g^{-1} g \qquad$ antipode

Definition

A $\Bbbk$-coalgebra is a triple (C, Δ, ε), where C is a $\Bbbk$-vector space $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \Bbbk$ are linear maps that satisfy the following commutative diagrams

<u>*Coassociativity*</u>*:*

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta\otimes \mathrm{id}"] \\ C \otimes C \arrow[r, "\mathrm{id}\otimes\Delta"'] & C \otimes C \otimes C \end{tikzcd}$$

<u>*Counit*</u>*:*

$$\begin{tikzcd} & C \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] \arrow[dd, "\Delta"'] & \\ \Bbbk \otimes C & & C \otimes \Bbbk \\ & C \otimes C \arrow[ul, "\varepsilon\otimes \mathrm{id}"] \arrow[ur, "\mathrm{id}\otimes\varepsilon"'] & \end{tikzcd}$$

Remark: The dual vector space C^* is an algebra (The converse is not always true).

Definition

A bialgebra is a $\Bbbk$-vector space B such that

- (B, m, u) *is an algebra;*
- (B, Δ, ε) *is a coalgebra;*
- Δ *and* ε *are algebra maps (*m *and* u *are coalgebra maps):*

$$\begin{tikzcd}
B \otimes B \arrow[r, "\Delta\otimes\Delta"] \arrow[dd, "m"'] & B \otimes B \otimes B \otimes B \arrow[d, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"] \\
 & B \otimes B \otimes B \otimes B \arrow[d, "m\otimes m"] \\
B \arrow[r, "\Delta"'] & B \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
B \otimes B \arrow[r, "\varepsilon\otimes\varepsilon"] \arrow[d, "m"'] & \Bbbk \otimes \Bbbk \arrow[d, "\simeq"] \\
B \arrow[r, "\varepsilon"'] & \Bbbk.
\end{tikzcd}$$

Example: $\mathcal{O}(M_n(\Bbbk))$ is a bialgebra.

Definition

A Hopf algebra is a $\mathbb{k}$-vector space H such that

- $(H, m, u, \Delta, \varepsilon)$ *is an bialgebra and*
- *there exists a map $\mathcal{S} : H \to H$ such that:*

$$
\begin{tikzcd}
 & H \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] \arrow[ddd, "u\varepsilon"] & \\
H \otimes H \arrow[d, "\mathrm{id} \otimes \mathcal{S}"'] & & H \otimes H \arrow[d, "\mathcal{S} \otimes \mathrm{id}"] \\
H \otimes H \arrow[dr, "m"'] & & H \otimes H \arrow[dl, "m"] \\
 & H &
\end{tikzcd}
$$

Example: $SL_n(\mathbb{k}) = \{A \in M_n(\mathbb{k}) \mid \det A = 1\}$

$$\mathcal{O}(SL_n(\mathbb{k})) = \mathbb{k}[X_{ij} \mid 1 \leq i, j \leq n]/(\det X - 1)$$

is the quotient of the commutative ring of polynomials in n^2 variables by the ideal generated by the element $\det X - 1$, where

$$\det X = \sum_{\sigma \in \mathbb{S}_n} (-1)^{\ell(\sigma)} X_{1\sigma(1)} X_{2\sigma(2)} \cdots X_{n\sigma(n)}$$

It is a Hopf algebra with

$$\Delta(X_{ij}) = \sum_{k=1}^{n} X_{ik} \otimes X_{kj}, \qquad \varepsilon(X_{ij}) = \delta_{ij} \qquad \mathcal{S}(X_{ij}) = \mathrm{Adj}(X)_{ji}$$

here $X = \left(X_{ij}\right)_{1 \leq i,j \leq n}$ and $\mathrm{Adj}(X)$ is the adjoint matrix.

For $n = 2$ we have

$$\mathcal{O}(SL_2(\Bbbk)) = \Bbbk[X_{11}, X_{12}, X_{21}, X_{22} | \ X_{11}X_{22} - X_{12}X_{21} = 1]$$

$$\Delta(X_{11}) = X_{11} \otimes X_{11} + X_{12} \otimes X_{21} \quad \varepsilon(X_{11}) = 1 \quad \mathcal{S}(X_{11}) = X_{22}$$

$$\Delta(X_{12}) = X_{11} \otimes X_{12} + X_{12} \otimes X_{22} \quad \varepsilon(X_{12}) = 0 \quad \mathcal{S}(X_{12}) = -X_{12}$$

$$\Delta(X_{21}) = X_{21} \otimes X_{11} + X_{22} \otimes X_{21} \quad \varepsilon(X_{21}) = 0 \quad \mathcal{S}(X_{21}) = -X_{21}$$

$$\Delta(X_{22}) = X_{21} \otimes X_{12} + X_{22} \otimes X_{22} \quad \varepsilon(X_{22}) = 1 \quad \mathcal{S}(X_{22}) = X_{11}$$

Following the same procedure as we did with $\mathcal{O}(M_n(\Bbbk))$ we obtain

$$\mathrm{Spec}(\mathcal{O}(SL_n(\Bbbk))) = \mathrm{Alg}_{\Bbbk}(\mathcal{O}(SL_n(\Bbbk)), \Bbbk) = SL_n(\Bbbk)$$

Theorem (Cartier)

Let H be a finitely generated commutative Hopf algebra.

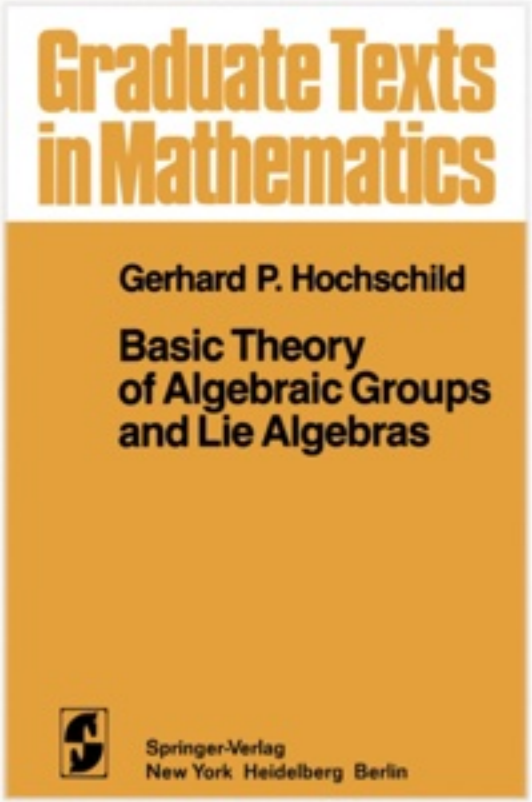
Cartier

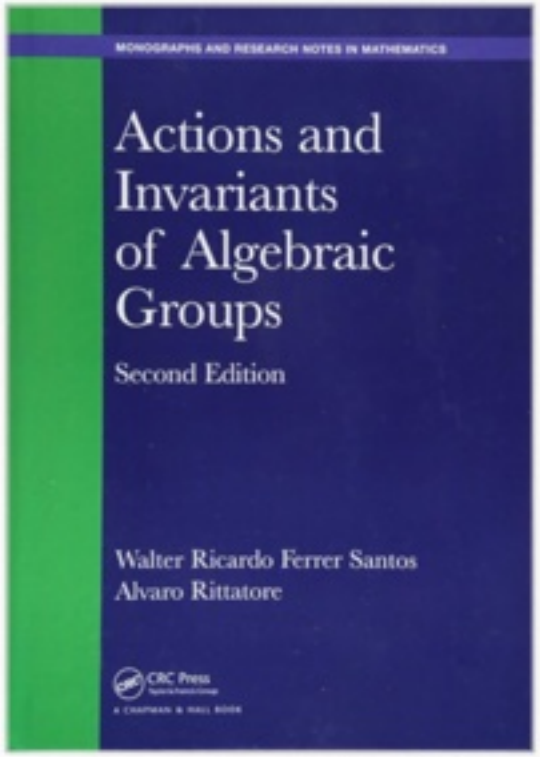
Then H is isomorphic to $\mathcal{O}(G)$ for some affine algebraic group G.

In categorical terms: There exists an equivalence of categories

$$\mathrm{AffGr}_{\Bbbk} \underset{\mathrm{Spec}}{\overset{\mathcal{O}}{\rightleftarrows}} \mathrm{FingenCommHa}_{\Bbbk}$$

This point of view can be found in several books, for example





Def Let C be a coalgebra.

We say that C is **cocommutative** if $\tau \circ \Delta = \Delta$; i.e.

$$\Delta(c) = c_{(1)} \otimes c_{(2)} = \tau \circ \Delta(c) = c_{(2)} \otimes c_{(1)}$$

Examples

1) $\Bbbk G$ group algebra of a group G

$$\Delta(g) = g \otimes g \ , \ \varepsilon(g) = 1$$

2) $U(\mathfrak{g})$ univ. envelop. alg. of Lie algebra $\mathfrak{g}$

$$\Delta(x) = x \otimes 1 + 1 \otimes x \ , \ \varepsilon(x) = -x \quad \forall x \in \mathfrak{g}$$

Theorem (Cartier-Gabriel-Kostant-Milnor-Moore)

Assume car $\mathbb{k}=0$, $\mathbb{k}=\overline{\mathbb{k}}$. If H is a cocommutative Hopf algebra, then

$$H \cong U(\mathfrak{g}) \# \mathbb{k}G \quad \text{with}$$

$$\mathfrak{g} = \{h \in H \mid \Delta(h) = h \otimes 1 + 1 \otimes h\}$$

$$G = \{h \in H \mid \Delta(h) = h \otimes h,\ h \neq 0\}$$

What is a quantum group?

$$\begin{array}{ccc} \text{Geometry} & \underset{\text{Spec}}{\overset{\mathcal{O}}{\rightleftarrows}} & \text{Algebra} \\ G & & \mathcal{O}(G) \\ & & \Big\downarrow \text{deformation} \\ ? & \longleftarrow & \mathcal{O}_q(G) \end{array}$$

Quantum Group $\longrightarrow$ NON-COMMUTATIVE GEOMETRY

NON-COMMUTATIVE ALGEBRA

Drinfeld (~'86): The category of quantum groups is the opposite category of Hopf algebras

What about cocommutative Hopf alg?

$$U(\mathfrak{g}) \xrightarrow[\text{deformation}]{\rightsquigarrow} U_q(\mathfrak{g})$$

NON-COCOMM.
Hopf algebra

Folklore: A quantum group is a non-commutative or non-cocommutative Hopf alg.

(Our) Definition

A **quantum group** is a deformation of an algebraic object that encodes a geometrical object describing symmetries, such as an algebraic group G or a Lie algebra $\mathfrak{g}$

Rmk Theory based on examples. We will see in this course some:

$$\mathcal{O}_q(G) \quad \text{and} \quad U_q(\mathfrak{g})$$