An Introduction to Quantum Groups and Hopf Algebras

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Lecture 2/4: Hopf algebras and tensor categories

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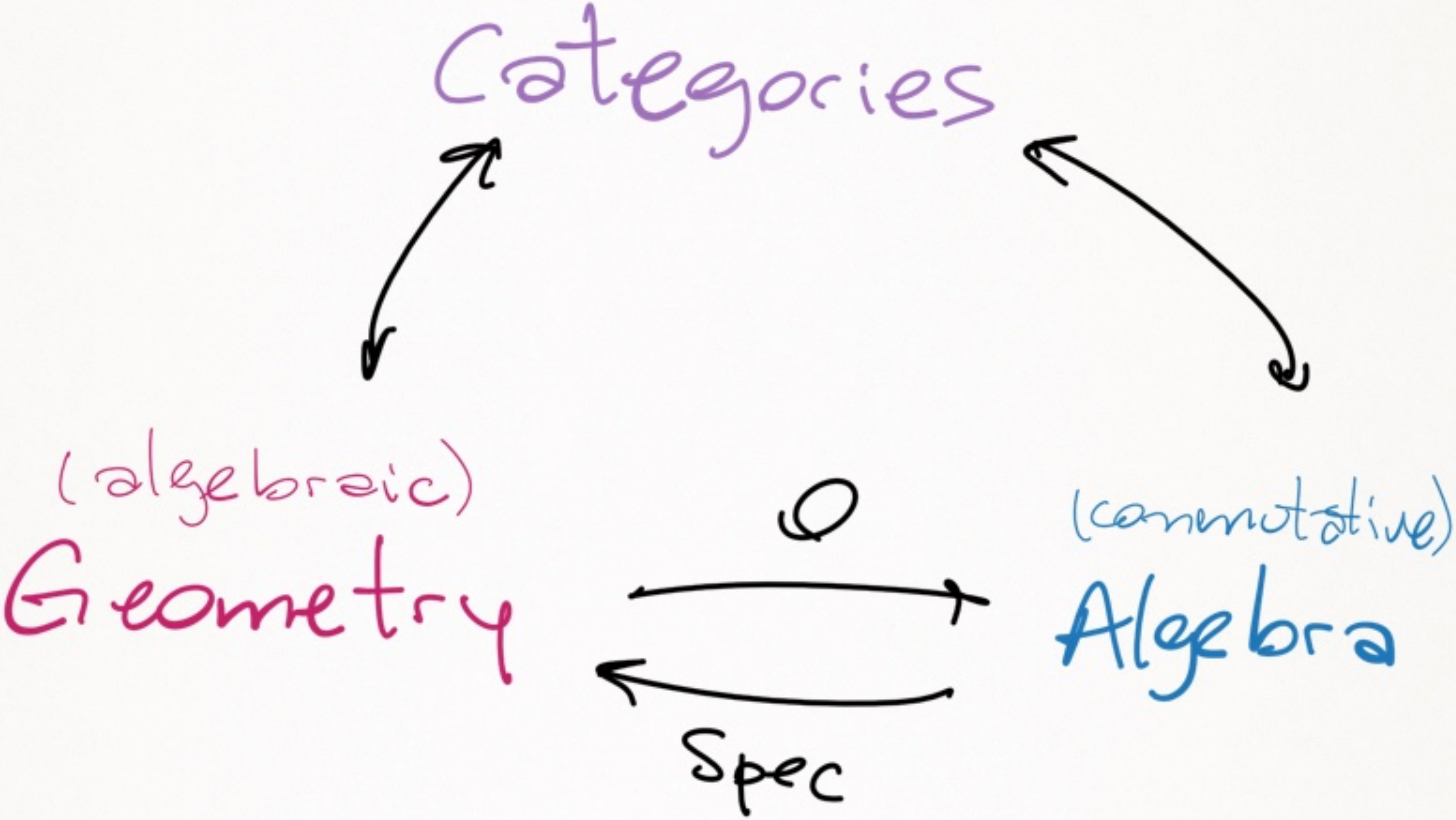
Reconstruction Theory


Categories
(algebraic)
Geometry
$\mathcal{O}$
(commutative)
Algebra
Spec

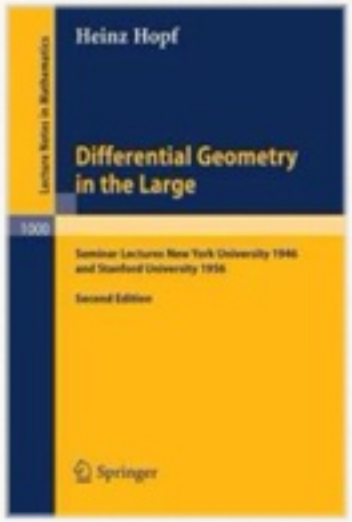
Algebra $\longleftrightarrow$ Categories

Hopf algebras

Heinz Hopf (1894-1971)

German mathematitian

Worked in topology and differential geometry



Definition

A $\Bbbk$-coalgebra is a triple (C, Δ, ε), where C is a $\Bbbk$-vector space $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \Bbbk$ are linear maps that satisfy the following commutative diagrams

<u>Coassociativity</u>:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta\otimes\mathrm{id}"] \\ C \otimes C \arrow[r, "\mathrm{id}\otimes\Delta"'] & C \otimes C \otimes C \end{tikzcd}$$

<u>Counit</u>:

$$\begin{tikzcd} & C \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] \arrow[dd, "\Delta"'] & \\ \Bbbk \otimes C & & C \otimes \Bbbk \\ & C \otimes C \arrow[ul, "\varepsilon\otimes\mathrm{id}"] \arrow[ur, "\mathrm{id}\otimes\varepsilon"'] & \end{tikzcd}$$

Definition

A Hopf algebra is an algebra (H, m, u) *and a coalgebra* (H, Δ, ε) *such that*

- Δ *and* ε *are algebra maps and*
- *there exists a linear map* $\mathcal{S} : H \to H$ *such that:*

$$
\begin{tikzcd}
 & H \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] \arrow[ddd, "u\varepsilon"] & \\
H \otimes H \arrow[d, "\mathrm{id} \otimes \mathcal{S}"'] & & H \otimes H \arrow[d, "\mathcal{S} \otimes \mathrm{id}"] \\
H \otimes H \arrow[dr, "m"'] & & H \otimes H \arrow[dl, "m"] \\
 & H &
\end{tikzcd}
$$

Remark

Let H be a finite dimensional Hopf algebra. Then H^* is also a Hopf algebra

$$(H, m, \eta, \Delta, \varepsilon, S) \rightsquigarrow (H^*, \Delta^*, \varepsilon^*, m^*, \eta^*, S^*)$$

$$H \otimes H \xrightarrow{m} H \qquad\qquad H^* \xrightarrow{m^*} (H \otimes H)^* \simeq H^* \otimes H^*$$

$$\Bbbk \xrightarrow{\eta} H \qquad\qquad H^* \xrightarrow{\eta^*} \Bbbk$$

$$H \xrightarrow{\Delta} H \otimes H \qquad\qquad H^* \otimes H^* \simeq (H \otimes H)^* \xrightarrow{\Delta^*} H^*$$

$$H \xrightarrow{\varepsilon} \Bbbk \qquad\qquad \Bbbk \xrightarrow{\varepsilon^*} H^*$$

$$H \xrightarrow{S} H \qquad\qquad H^* \xrightarrow{S^*} H^*$$

- $(H^*)^* \simeq H$

Examples

1) G a finite group, $\{g\}_{g\in G}$ linear basis

$$\Bbbk G = \{ \sum_{g\in G} \lambda_g g \mid \lambda_g \in \Bbbk \}$$

$$\Delta(g) = g\otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}$$

$$(\Bbbk G)^* = \mathcal{O}(G) = \Bbbk^G = \{ \alpha : G \to \Bbbk \mid \text{functions} \}$$

For the dual basis $\{\delta_g\}_{g\in G}$

$$\delta_g(h) = \delta_{gh} = \begin{cases} 1 & g=h \\ 0 & \text{oth.} \end{cases}$$

$$\Delta(\delta_g) = \sum_{h\in G} \delta_{gh^{-1}} \otimes \delta_h, \quad \varepsilon(\delta_g) = \delta_{g,e}, \quad S(\delta_g) = \delta_{g^{-1}}$$

Remark

- $\Bbbk G$ is cocommutative
- $\Bbbk G$ is commutative iff G is abelian
- $\Bbbk \widehat{G} \subseteq \mathcal{O}(G)$ and $\Bbbk \widehat{G} = \mathcal{O}(G)$ iff G is abelian

2) Sweedler algebra

$$H_4 = \Bbbk\langle g, x \mid g^2 = 1,\ x^2 = 0,\ gx = -xg\rangle$$

$$\Delta(g) = g \otimes g,\ \varepsilon(g) = 1,\ S(g) = g^{-1}$$

$$\Delta(x) = x \otimes 1 + g \otimes x,\ \varepsilon(x) = 0,\ S(x) = -gx$$

- It is non-commut. and non-cocomm.
- $\dim H_4 = 4$
- $H_4^* \cong H_4$
- H_4 is related to quantum groups

Representations

Def Let A be an algebra.
A *linear representation* of A is a pair (V,ρ) where

- V is a $\mathbb{k}$-vector space
- $\rho : A \longrightarrow \mathrm{End}(V)$ is an algebra map
  - $\rho(1) = \mathrm{id}$
  - $\rho(a+\lambda b) = \rho(a) + \lambda\rho(b)$
  - $\rho(ab) = \rho(a)\circ\rho(b) \quad \forall a,b\in A$

Remark A an algebra

(V, ρ) rep. of A $\longleftrightarrow$ V left A-module

$$\rho : A \longrightarrow \mathrm{End}(V), \quad a \longmapsto \rho(a)$$

$$A \otimes V \xrightarrow{\ \cdot\ } V, \quad a \otimes v \longmapsto a \cdot v$$

- $1 \cdot v = v$
- $(ab) \cdot v = a \cdot (b \cdot v)$

$$\rho(a) \longleftrightarrow a \cdot \_$$

$$\rho(ab) = \rho(a) \circ \rho(b)$$

$$\begin{tikzcd} A \otimes A \otimes V \arrow[r, "\mathrm{id} \otimes \cdot"] \arrow[d, "m \otimes \mathrm{id}"'] \arrow[dr, phantom, "\circlearrowright"] & A \otimes V \arrow[d, "\cdot"] \\ A \otimes V \arrow[r, "\cdot"'] & V \end{tikzcd}$$

Def A morphism between two rep. (V_1, ρ_1), (V_2, ρ_2) is a linear map

$$V_1 \xrightarrow{\varphi} V_2$$ such that

$$\begin{tikzcd}
V_1 \arrow[r, "\varphi"] \arrow[d, "\rho_1(a)"'] \arrow[dr, phantom, "\circlearrowright"] & V_2 \arrow[d, "\rho_2(a)"] \\
V_1 \arrow[r] & V_2
\end{tikzcd} \qquad \forall a \in A$$

$\mathrm{Rep}(A)$ = category of representations of A

$= {}_A\mathcal{M}$ (cat of left A-mod)

$\mathrm{rep}(A)$ = category of finite-dimensional rep. of A

Def Let C be a coalgebra.

A **corepresentation or a left C-comodule** is (V, λ) where

- V is a $\mathbb{k}$-vector space
- $\lambda : V \longrightarrow C \otimes V$ is a linear map s.t

$$\begin{tikzcd} V \arrow[r, "\lambda"] \arrow[d, "\lambda"'] \arrow[dr, phantom, "\circlearrowright"] & C\otimes V \arrow[d, "\Delta\otimes id"] \\ C\otimes V \arrow[r, "id\otimes\lambda"'] & C\otimes C\otimes V \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\lambda"] \arrow[dr, equal, "id"'] & C\otimes V \arrow[d, "\varepsilon\otimes id"] \\ & V \end{tikzcd}$$

${}^{C}\mathcal{M} = C\text{-comod} =$ cat. of left C-comod

Proposition Let H be a Hopf algebra

1) $V, W \in \mathrm{Rep}\, H \Rightarrow V \otimes W \in \mathrm{Rep}\, H$

2) $\mathbb{k} \in \mathrm{Rep}\, H$

3) $V \in \mathrm{Rep}\, H \Rightarrow V^* \in \mathrm{Rep}\, H$

Remark Assume S is invertible.
Then $\mathrm{Rep}\, H$ is a tensor category:

An analogous proposition is true for C-comod

Definition (EGNO)

A *tensor category* $\mathcal{C}$ *is a*

- *monoidal category (has bifunctor* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$*);*
- *abelian (has* $\oplus$*, kernels and cokernels);*
- $\Bbbk$*-linear (*$\mathrm{Hom}_{\mathcal{C}}(X, Y)$ *is* $\Bbbk$*-v.s. and* $\circ$ *is linear map);*
- *locally finite (*$\dim \mathrm{Hom}_{\mathcal{C}}(X, Y) < \infty$ *and* $\ell(X) < \infty$*);*
- *rigid (has duals);*
- $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) = \Bbbk$.

__Question:__ Given a tensor category $\mathcal{C}$

There exists a Hopf algebra H such that $\mathcal{C} \simeq \mathrm{Rep}\, H$?

If $\mathcal{C}$ is finite, yes!!

__Idea:__ relate $\mathcal{C}$ with Vec through a nice functor

Etingof
Gelaki
Nykshych
Ostrik

Definition (EGNO)

Let $\mathcal{C}$ be a tensor category. A fiber functor is an exact faithful functor

$$F : \mathcal{C} \to \mathrm{Vec}$$

such that $F(\mathbf{1}) = \Bbbk$ and there is a natural transformation

$$J : F(X) \otimes F(Y) \to F(X \otimes Y) \qquad \forall\ X, Y \in \mathcal{C}$$

which is a tensor structure:

$$\begin{tikzcd}
F(X) \otimes F(Y) \otimes F(Z) \arrow[rr] \arrow[d, "J_{X,Y} \otimes \mathrm{id}"'] & & F(X) \otimes (F(Y) \otimes F(Z)) \arrow[d, "\mathrm{id} \otimes J_{Y,Z}"] \\
F(X \otimes Y) \otimes F(Z) \arrow[d, "J_{X \otimes Y, Z}"'] & \circlearrowleft & F(X) \otimes (F(Y \otimes Z)) \arrow[d, "J_{X, Y \otimes Z}"] \\
F((X \otimes Y) \otimes Z) \arrow[rr] & & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

Example

Let H be a Hopf algebra.

Then the forgetful functor

$$\begin{aligned} \text{Forget}: \operatorname{rep} H &\longrightarrow \mathrm{Vec} \\ (V, \rho) &\longmapsto V \end{aligned}$$

is a fiber functor

Let $\mathcal{C}$ be a finite tensor category
($\mathcal{C} \simeq \mathrm{rep}(A)$ for some A fin-dim alg)
and $F: \mathcal{C} \longrightarrow \mathrm{Vec}$ a fiber functor

$\Rightarrow$ $\mathrm{End}(F) =: H$ is a unital associative algebra with $1 = \mathrm{id}$ and composition

Theorem $H = \mathrm{End}(F)$ is a finite-dimensional Hopf algebra

Theorem (Tannaka-Krein reconstruction)

There is a bijection

Finite tensor cat with fiber functors (up to tensor equiv & iso)	Finite-dimensional Hopf algebras (up to iso)
$(\mathcal{C}, F) \longrightarrow$	$\mathrm{End}(F)$
$(\mathrm{rep}\, H, \mathrm{Forget}) \longleftarrow$	H

What can be said in the **non-finite** setting?

First issue: If H is an infinite dimensional Hopf algebra, then H^* is not a Hopf algebra in general

look for such a deal inside H^*

Def Let H be an algebra

The finite or restricted dual is the subspace of H^* given by

$$H^\circ = \{ f \in H^* \mid f(I) = 0 \text{ for some ideal } I \subseteq H,\ \dim H/I < \infty \}$$

Lemma (H°, m^*, u^*) is a coalgebra

Theorem If H is a Hopf algebra then H° is a Hopf algebra

Examples

1) G simply connected affine alg. group. $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra (e.g. $G = SL_2(\mathbb{C})$, $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$)

$\Rightarrow U(\mathfrak{g})^\circ \cong \mathcal{O}(G)$

But $U(\mathfrak{g}) \hookrightarrow \mathcal{O}(G)^\circ$ in general

2) If $\dim H < \infty$, then $H^\circ = H^*$

Constructing H° via representations

Let H be a Hopf algebra
$V \in \operatorname{rep} H$

For $v \in V$, $f \in V^*$ define the **matrix coefficient**

$$c^V_{f,v} : H \longrightarrow \Bbbk \quad \text{by}$$

$$c^V_{f,v}(h) = f(h \cdot v) \quad \forall\, h \in H$$

Remark We have a map

$$c^V : V^* \otimes V \longrightarrow H^*$$

1) It is the dual map of

$$\rho : H \longrightarrow \mathrm{End}(V) \simeq V^* \otimes V$$

2) $\mathrm{Im}\, c^V \subseteq H^\circ$

3) $\mathrm{Im}\, c^V$ is a subcoalgebra of H°

Proposition $H^\circ = \sum_{V \in \mathrm{rep} H} \mathrm{Im}\, c^V$

i.e. H° is the span of matrix coeff. of finite-dimensional representations of H
$\longleftrightarrow$ fin-dim corepresentations of H°

Writting $\mathrm{Im}\, c^V \cong V^* \otimes V$ we obtain a Peter-Weyl version

$$\boxed{H^\circ = \sum_{V \in \mathrm{rep} H} V^* \otimes V}$$

The Coend Construction

Def Let $\mathcal{C}$ be a tensor category $F : \mathcal{C} \longrightarrow Vec$ an exact, faithful Functor. Then

$$Coend(F) = \bigoplus_{X \in \mathcal{C}} F(X)^* \otimes F(X) \Big/ E$$

E is spanned by the elements

$$y_* \otimes F(f)x - F(f)^* y_* \otimes x \quad , \quad \begin{array}{l} x \in F(X) \\ y \in F(Y)^* \\ f \in Hom(X,Y) \end{array}$$

It holds

- $\mathrm{Coend}(F) = \varinjlim \mathrm{End}(F(X))^*$
- $\mathrm{Coend}(F)^* = \mathrm{End}(F)$
- $\mathrm{Coend}(F)$ is a coalgebra

<u>Prop</u> $\mathcal{C}$ tensor category with a fiber functor $F: \mathcal{C} \to \mathrm{Vec}$

Then $\mathrm{Coend}(F)$ is a Hopf algebra

<u>proof</u> see [EGNO, Tensor Categories]

Theorem (Reconstruction Theorem)

There is a bijective correspondance

$$\left\{\begin{array}{c}\text{Tensor categories}\\ \text{with a fiber functor}\\ \text{(up to equiv \& iso)}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Hopf algebras}\\ \text{(up to iso)}\end{array}\right\}$$

$$(\mathcal{C}, F) \longrightarrow \mathrm{Coend}(F)$$

$$(H\text{-comod}, \mathrm{Forget}) \longleftarrow H$$

Example

G affine algebraic group

$\mathcal{C}$ = category of algebraic representations

$F : \mathcal{C} \longrightarrow \mathrm{Vec}$ Forgetful functor

$$\Rightarrow \quad \mathrm{Coend}(F) = \mathcal{O}(G)$$

algebra of rational functions

Summary

$$\begin{tikzcd}
\text{Hopf algebras} \arrow[r, bend left, "\text{rep, comod}"] & \text{Tensor cat w/ fibres functors} \arrow[l, bend left, "\text{Tannaka-Krein}"]
\end{tikzcd}$$

$$\begin{tikzcd}
H \arrow[r, "\dim H<\infty"] \arrow[dr] & (\mathrm{rep}\, H, \mathrm{Forget}) \\
& (H\text{-comod}, \mathrm{Forget})
\end{tikzcd}$$

$$\begin{tikzcd}
H = \mathrm{End}(F) & (\mathcal{C}, F) \arrow[l, "\mathcal{C}\ \text{finite}"'] \arrow[dl] \\
H^{\circ} = \mathrm{Coend}(F) &
\end{tikzcd}$$