# An Introduction to Quantum Groups and Hopf Algebras

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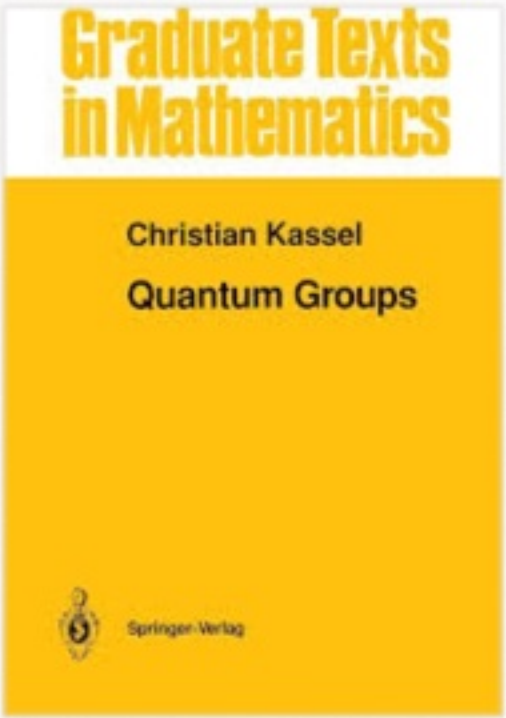
Lecture 3/4: Quantum coordinate algebras

*ICTP, Trieste*

October 14 – 27, 2024

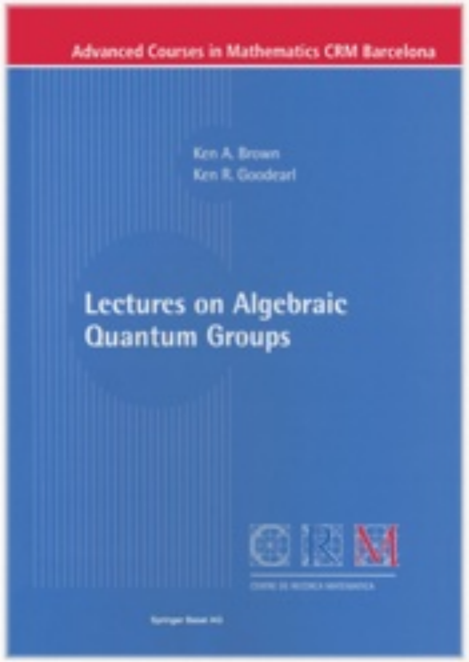
# The Quantum Setting

For more details see (for example)




Kassel


Parshall
Wang




Brown
Goodearl

## Definition ($M_q(n)$)

*Let $n \in \mathbb{N}$ and $q \in \Bbbk - \{0\}$. $\mathcal{O}_q(M_n(\Bbbk))$ is the unital associative algebra over $\Bbbk$ generated by $n^2$ elements $X_{ij}$, $1 \leq i, j \leq n$ such that:*

$$\begin{aligned} X_{ri}X_{rj} &= qX_{rj}X_{ri} \quad \text{if } i < j; \\ X_{is}X_{js} &= qX_{js}X_{is} \quad \text{if } i < j; \\ X_{ri}X_{sj} &= X_{sj}X_{ri} \quad \text{if } r < s \text{ and } i > j; \\ X_{ri}X_{sj} - X_{sj}X_{ri} &= (q - q^{-1})X_{si}X_{rj} \quad \text{si } r < s \text{ and } i < j. \end{aligned}$$

**Remark** For $q = 1$ we have $\mathcal{O}_1(M_n(\Bbbk)) = \mathcal{O}(M_n(\Bbbk))$.

## Proposition

*$\mathcal{O}_q(M_n(\mathbb{k}))$ is a non-commutative and non-cocommutative bialgebra with*

$$\Delta(X_{ij}) = \sum_{k=1}^{n} X_{ik} \otimes X_{kj}, \qquad \varepsilon(X_{ij}) = \delta_{ij} \qquad \textit{for all } 1 \le i,j \le n.$$

Let us check the equality

$$\Delta(x_{ir}x_{jr}) = \Delta(q\, x_{jr}x_{ir}) \quad \text{for } \boxed{i<j}$$

$$\Delta(x_{ir}x_{jr}) \underset{\Delta \text{ alg. map}}{=} \Delta(x_{ir})\Delta(x_{jr}) =$$

$$= \left(\sum_{k=1}^{n} x_{ik}\otimes x_{kr}\right)\left(\sum_{l=1}^{n} x_{jl}\otimes x_{lr}\right)$$

$$= \sum_{k,l=1}^{n} x_{ik}x_{jl}\otimes x_{kr}x_{lr}$$

On the other hand

$$\Delta(qX_{js}X_{ir}) = q\,\Delta(X_{jr})\Delta(X_{ir}) =$$

$$= q\left(\sum_{\ell=1}^{n} X_{j\ell}\otimes X_{\ell r}\right)\left(\sum_{k=1}^{n} X_{ik}\otimes X_{kr}\right)$$

$$= q\left(\sum_{\ell,k=1}^{n} X_{j\ell}X_{ik}\otimes X_{\ell r}X_{ks}\right)$$

$$= q\left(\sum_{\ell<k} X_{j\ell}X_{ik}\otimes X_{\ell s}X_{ks} +\right.$$

$$+\sum_{\ell=k} X_{j\ell}X_{i\ell}\otimes X_{\ell s}X_{\ell r}$$

Equation 4 with $i<j$ and $\ell>k$

$$\left.+\sum_{\ell>k} X_{j\ell}X_{ik}\otimes X_{\ell s}X_{ks}\right) =$$

$$= q\Big(\sum_{l>k} X_{jl}X_{ik}\otimes X_{lr}X_{ks} + \sum_{l=k} X_{jl}X_{il}\otimes X_{lr}X_{ls}$$

$$+ \sum_{l<k}\big(X_{ik}X_{jl} - (q-q^{-1})X_{jk}X_{il}\big)\otimes X_{lr}X_{kr}$$

$$= q\Big(\sum_{l>k} X_{jl}X_{ik}\otimes X_{lr}X_{kr} + \sum_{l=k} q^{-1}X_{il}X_{jl}\otimes X_{lr}X_{ls}$$

Equation 2

$$+ \sum_{l<k} X_{ik}X_{jl}\otimes X_{lr}X_{kr} - q\sum_{l<k} X_{jk}X_{il}\otimes X_{lr}X_{ks}$$

$$+ q^{-1}\sum_{l<k} X_{jk}X_{il}\otimes X_{lr}X_{ks}$$

COMMUTE!

(Equation 3)

We are using all equations!!

$$= q\Big( \cancel{\sum_{l<k} x_{jl} x_{ik} \otimes x_{lr} x_{kr}} + q^{-1} \sum_{l=k} x_{il} x_{jl} \otimes x_{lr} x_{lr}$$

$$+ \sum_{l>k} x_{ik} x_{jl} \otimes x_{lr} x_{kr} - \cancel{q \sum_{l>k} x_{jk} x_{il} \otimes q^{-1} x_{kr} x_{lr}}$$

$$+ q^{-1} \sum_{l<k} x_{il} x_{jk} \otimes x_{lr} x_{kr} \Big)$$

$$\overset{\text{Eq.2}}{=} q\Big( q^{-1} \sum_{l=k} x_{il} x_{jl} \otimes x_{lr} x_{lr} + \sum_{l>k} x_{ik} x_{jl} \otimes q^{-1} x_{kr} x_{lr}$$

$$+ q^{-1} \sum_{l<k} x_{il} x_{jk} \otimes x_{lr} x_{kr}$$

$$= \sum_{l,k=1}^{n} x_{il} x_{jk} \otimes x_{lr} x_{jr} \quad ✓$$

## Proposition

*$\mathcal{O}_q(M_n(\Bbbk))$ is a non-commutative and non-cocommutative bialgebra with*

$$\Delta(X_{ij}) = \sum_{k=1}^{n} X_{ik} \otimes X_{kj}, \qquad \varepsilon(X_{ij}) = \delta_{ij} \qquad \textit{for all } 1 \leq i,j \leq n.$$

**Remark** $\mathcal{O}_q(M_n(\Bbbk))$ is not a Hopf algebra $\rightsquigarrow$ there exists a group like element, the *quantum determinant*.

## Proposition (The quantum determinant)

$$\det_q X = \sum_{\sigma \in \mathbb{S}_n} (-q)^{\ell(\sigma)} X_{\sigma(1),1} \cdots X_{\sigma(n),n} \qquad \in \mathcal{O}_q(M_n(\Bbbk))$$

*is a central group-like element ($\ell(\sigma)$ is the length of the permutation $\sigma$).*

## Definition (The quantum groups $GL_q(n)$ and $SL_q(n)$ )

$$\mathcal{O}_q(GL_n(\Bbbk)) = \mathcal{O}_q(M_n(\Bbbk))[T]/(T\det_q X - 1)$$

$$\mathcal{O}_q(SL_n(\Bbbk)) = \mathcal{O}_q(M_n(\Bbbk))/(\det_q X - 1)$$

**Theorem** $\mathcal{O}_q(GL_n(\Bbbk))$ and $\mathcal{O}_q(SL_n(\Bbbk))$ are Hopf algebras

Pf The two-sided ideals generated by $T\det_q X - 1$ and $\det_q X - 1$ are bi-ideals of $\mathcal{O}_q(M_n(\Bbbk))$

$\rightsquigarrow$ define antipode using $\det_q X$ ◼

For $n=2$, $\mathcal{O}_q(GL_2(\Bbbk))$

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = T\begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix} = (\det_q X)^{-1}\begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}$$

<u>Question</u>: Are the Hopf algebras $\mathcal{O}_q(G)$ and $\mathcal{O}(G)$ related?

If $q$ is a root of unity, there is a strong relation besides $\mathcal{O}_1(G) = \mathcal{O}(G)$

# Specializations at roots of 1 and the quantum Frobenius Map

Let $\ell \in \mathbb{N}$ odd, $q^{\ell}=1$ primitive

**Lemma**

1) $X_{ij}^{\ell}$ is central in $\mathcal{O}_q(M_n(\mathbb{k}))$

2) $\Delta(X_{ij}^{\ell}) = \sum_{k=1}^{n} X_{ik}^{\ell} \otimes X_{kj}^{\ell}$, $\epsilon(X_{ij}^{\ell}) = \delta_{ij}$

**Def** The quantum Frobenius map is the map

$$\mathcal{O}(M_n(\mathbb{k})) \xrightarrow{F} \mathcal{O}_q(M_n(\mathbb{k}))$$

$$X_{ij} \longmapsto X_{ij}^{\ell} \quad \forall i,j$$

**Prop** 1) $F$ is a bialgebra monomorph.

2) $F(\det X) = (\det_q X)^{\ell}$

The quantum Frobenius map for $M_n(\mathbb{k})$ induces the quantum Frobenius maps

$$\mathcal{O}(GL_n(\mathbb{k})) \xhookrightarrow{F} \mathcal{O}_q(GL_n(\mathbb{k}))$$

$$\mathcal{O}(SL_n(\mathbb{k})) \xhookrightarrow{F} \mathcal{O}_q(SL_n(\mathbb{k}))$$

This holds for $G$ more general.

Moreover, we have a s.e.c of Hopf algebras

$$\mathcal{O}(G) \hookrightarrow \mathcal{O}_q(G) \longrightarrow A$$

where $\dim A = q^{\dim G}$ ; and

$$\mathcal{O}_q(G) \cong \mathcal{O}(G) \otimes A$$

as left $\mathcal{O}(G)$-modules and $A$-comodules

# The FRT-Construction

Fadeev

Reshetikhin

Takhtajan

St. Petersburg School

<u>Idea</u>: Use the symmetry

$$V \otimes W \cong W \otimes V$$

in $\mathrm{Rep}\,H$ to construct the $\mathrm{Coend}(F)$

- $V$ should be a left comodule
- $V \otimes V \xrightarrow{c} V \otimes V$ a comodule map

Let $V$ be a finite-dim. vector space

$\{v_1, v_2, \ldots, v_n\}$ a basis

$c \in \mathrm{End}(V \otimes V)$ $\longrightarrow$ $\{v_i \otimes v_j\}_{1 \le i,j \le n}$ basis of $V \otimes V$

$$c(v_i \otimes v_j) = \sum_{k,l} c_{ij}^{kl} \, v_k \otimes v_l$$

## Take

- $C$ vector space with basis $\{x_{ij}\}_{1\leq i,j\leq n}$

$\Rightarrow$ $C$ is a coalgebra with

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}, \quad \varepsilon(x_{ij}) = \delta_{ij}$$

$\forall 1 \leq i,j \leq n$

- $V$ is a left $C$-comodule with

$$\lambda(v_i) = \sum_{j=1}^{n} x_{ij} \otimes v_j \quad \forall 1 \leq i \leq n$$

- $TC$ = free unital associative algebra over $\mathbb{C}$
  = polynomial ring in non-commutative variables $\{x_{ij}\}$

- $TC$ is a coalgebra by declaring $\Delta$ and $\varepsilon$ to be algebra maps

- $V \otimes V$ is a left $TC$-comodule

$$\lambda(v_i \otimes v_j) = \sum_{k,l=1}^{n} \underbrace{x_{ik} x_{jl}}_{\in TC} \otimes v_k \otimes v_l$$

$$
\begin{tikzcd}
v_i \otimes v_j \arrow[rrr, mapsto, bend left] \arrow[d, mapsto] & V \otimes V \arrow[r, "c"] \arrow[d, "\lambda"] & V \otimes V \arrow[d, "\lambda"] & \sum_{k,\ell} c_{ij}^{k\ell} v_k \otimes v_\ell \arrow[d, mapsto] \\
\sum_{k,\ell} x_{ik} x_{j\ell} \otimes v_k \otimes v_\ell \arrow[drrr, mapsto] & TC \otimes (V \otimes V) \arrow[r, "\mathrm{id} \otimes c"] & TC \otimes (V \otimes V) & \sum_{r,s,k,\ell} c_{ij}^{k\ell} \, x_{kr} x_{\ell s} \otimes v_r \otimes v_s \\
 & & & \sum_{k,\ell,r,s} x_{ik} x_{j\ell} c_{k\ell}^{rs} \otimes v_r \otimes v_s
\end{tikzcd}
$$

## Definition (FRT)

*The FRT-construction (or universal quantum semigroup) for* $(V, c)$ *is the* $\Bbbk$*-algebra* $A = A(c)$ *generated by the elements* $\{x_{ij}\}_{1 \le i,j \le n}$*, satisfying the following relations:*

$$\boxed{\boxed{\sum_{k,\ell} c_{ij}^{k\ell} x_{kr} x_{\ell s} = \sum_{k,\ell} x_{ik} x_{j\ell} c_{k\ell}^{rs}}} \qquad \forall\ 1 \le i, j, r, s \le n.$$

**Remark** The construction works for any $c \in \mathrm{End}(V \otimes V)$. But one usually looks for solutions of the braid equation:

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c) \qquad \text{in } \mathrm{End}(V \otimes V \otimes V).$$

## Theorem

*$A(c)$ is a bialgebra that satisfies the following properties*

(a) *$V$ is a left $A(c)$-comodule;*

(b) *$c$ is a $A(c)$-comodule map;*

(c) *[U.P.] If $B$ is A bialgebra such that $V$ is a left $B$-comodule and $c$ is a $B$-comodule map, then there exists a unique bialgebra map $f : A(c) \to B$ such that*

$$\begin{tikzcd} V \arrow[r, "\delta_V"] \arrow[dr, "\delta'_V"'] & A(c) \otimes V \arrow[d, dashed, "f \otimes \mathrm{id}"] \\ & B \otimes V. \end{tikzcd}$$

*In particular, the bialgebra $A(c)$ is unique up to isomorphism.*

**Example:** Let $V$ be a $\mathbb{k}$-vector space of dimension $n$ and basis $B=\{v_1,\dots,v_n\}$.

Let $q\in\mathbb{k}-\{0\}$ and consider $c: V\otimes V\to V\otimes V$ given by

- $c(v_i\otimes v_i)=q^{-1}v_i\otimes v_i$ for all $1\leq i\leq n$,
- $c(v_i\otimes v_j)=\begin{cases} v_j\otimes v_i & \text{si } i<j,\\ v_j\otimes v_i+(q^{-1}-q)v_i\otimes v_j & \text{if } i>j.\end{cases}$

For $n=2$ and basis $B_{V\otimes V}=\{v_1\otimes v_1, v_1\otimes v_2, v_2\otimes v_1, v_2\otimes v_2\}$:

$$[c]=\begin{pmatrix} q^{-1} & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & q^{-1}-q & 0\\ 0 & 0 & 0 & q^{-1}\end{pmatrix}.$$

**Theorem**

$A(c)\simeq\mathcal{O}_q(M_n(\mathbb{k}))$

**Question:** Is it possible to construct a quantum group (Hopf alg) from the bialgebra $A(c)$?

**Theorem** (Schauenburg)

If $c \in \mathrm{Aut}(V \otimes V)$ is a rigid solution of the braid equation, then there exists a universal Hopf algebra $H(c)$

**Question:** Can we obtain the quantum group $H(c)$ by using a quantum det?

Yes, under some conditions on $c$.

Answer in a joint work with M. Farinati [Quantum function algebras from finite-dimensional Nichols alg]

<u>Idea</u>: Take $V$, $\dim V = n$
$c \in \operatorname{End}(V \otimes V)$ a braiding

1) Define a quantum det. $\mathcal{D}$ for $A(c)$
2) Prove $\mathcal{H}(c) = A(c)[\mathcal{D}^{-1}]$ is a Hopf alg
3) Prove $\mathcal{H}(c) = H(c)$

# Quantum determinants

<u>Classical case:</u> $M_n(\Bbbk)$ acts on $V \cong \Bbbk^n$

$\Updownarrow$

$\mathcal{O}(M_n(\Bbbk))$ coacts on $V$

$B = \{v_1, \dots, v_n\}$ basis of $V$

$$\lambda : V \longrightarrow \mathcal{O}(M_n(\Bbbk)) \otimes V$$

$$\lambda(v_i) = \sum_{j=1}^{n} x_{ij} \otimes v_j \quad \forall i$$

This coaction can be extended to $T(V)$:

$$\lambda(v_{i_1}\otimes\cdots\otimes v_{i_s}) = \sum_{j_1,\ldots,j_s} x_{i_1j_1}\cdots x_{i_sj_s}\otimes(v_{j_1}\otimes\cdots\otimes v_{j_s})$$

<u>Note</u>: $T(V) = \bigoplus_{k\geq 0} V^{\otimes k} = \Bbbk\oplus V\oplus(V\otimes V)\oplus\cdots$

Each $V^{\otimes k}$ is a $\mathcal{O}(M_n(\Bbbk))$-comodule

Also, $J = (v_i\otimes v_j + v_j\otimes v_i)$ is a subcomodule

$\Longrightarrow$ $T(V)/J = \Lambda V$ is a $\mathcal{O}(M_n(\Bbbk))$-comod

Since $\Lambda^n V = \mathbb{k}\, v_1 \wedge v_2 \wedge \cdots \wedge v_n$

$\Rightarrow \lambda(v_1 \wedge v_2 \wedge \cdots \wedge v_n) = d \otimes v_1 \wedge v_2 \wedge \cdots v_n$

$v = v_1 \wedge v_2 \wedge \cdots \wedge v_n$ volume element

$d = \det X$

For $A(c)$, find replacement for $\underline{\underline{\Lambda^n V}}$

## Definition

Let $\mathcal{A}$ be a bialgebra and $V \in {}^{\mathcal{A}}\mathcal{M}$. An $\mathcal{A}$-comodule algebra $\mathcal{B}$ is called a *weakly graded-Frobenius* algebra for $\mathcal{A}$ and $V$ if the following conditions are satisfied:

- $\mathcal{B}$ is an $\mathbb{N}$-graded $\mathcal{A}$-comodule algebra, that is $\mathcal{B} = \bigoplus_{n \geq 0} \mathcal{B}^n$, $\lambda(\mathcal{B}^n) \subseteq \mathcal{A} \otimes \mathcal{B}^n$ and $\mathcal{B}^n \cdot \mathcal{B}^m \subseteq \mathcal{B}^{n+m}$ for all $n, m \geq 0$;
- $\mathcal{B}^0 = \Bbbk$ and $\mathcal{B}^1 = V$ as $\mathcal{A}$-comodules;
- $\dim_{\Bbbk} \mathcal{B} < \infty$ and $\dim_{\Bbbk} \mathcal{B}^{top} = 1$, where $top = \max\{n \in \mathbb{N} : \mathcal{B}^n \neq 0\}$;
- the multiplication induces non-degenerate bilinear maps

$$\mathcal{B}^1 \times \mathcal{B}^{top-1} \to \mathcal{B}^{top}, \qquad \mathcal{B}^{top-1} \times \mathcal{B}^1 \to \mathcal{B}^{top}.$$

## Definition

Let $\mathcal{B}$ be a weakly graded-Frobenius algebra for $A$ and $\mathcal{B}^{top} = \Bbbk v$.

- $v$ is a *volume element* for $\mathcal{B}$.
- the element $D$ such that $\lambda(v) = D \otimes v$ is the *quantum determinant* in $A$ associated with $\mathcal{B}$.

**Remarks**

$(a)$ $D \in G(A)$ is independent of the scalar multiple of $v$.

$(b)$ Nichols algebras associated to a braiding $c$ are a source of examples (in case they are finite-dimensional).

**Example:** $X = \{1,2\}$ and $s : X \times X \to X \times X$ (set) braiding

$s(1,2) = (1,2), \quad s(2,1) = (2,1), \quad s(1,1) = (2,2), \quad s(2,2) = (1,1).$

If $(x_{ij})_{i,j} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then $A(-s)$ has relations

$$a^2 = d^2, \quad ab = cd, \quad ba = dc, \quad ac = bd, \quad ca = db, \quad b^2 = c^2.$$

Nichols algebra $\mathcal{B} = \mathcal{B}(V, -s)$ with $V = \Bbbk x \oplus \Bbbk y$ has relations

$$x^2 + y^2 = 0, \qquad 2xy = 0 = 2yx.$$

If $\mathrm{char}(\Bbbk) \neq 2$, then $\dim \mathcal{B} < \infty$ and $\mathcal{B}$ has basis $\{1, x, y, x^2\}$.
Volume element $v = x^2$.
Quantum determinant $D := a^2 - b^2$ (is central).
$H(s) = A[D^{-1}] =: \mathbf{GL}(X, -s)$ Hopf algebra with antipode

$$\mathcal{S}(a) = aD^{-1}, \ \mathcal{S}(b) = -cD^{-1}, \ \mathcal{S}(c) = -bD^{-1}, \ \mathcal{S}(d) = dD^{-1}.$$

Also $\mathbf{SL}(X, -s) = A(s)/(D-1)$.