An Introduction to Quantum Groups and Hopf Algebras

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Lecture 4/4: Universal quantum enveloping algebras

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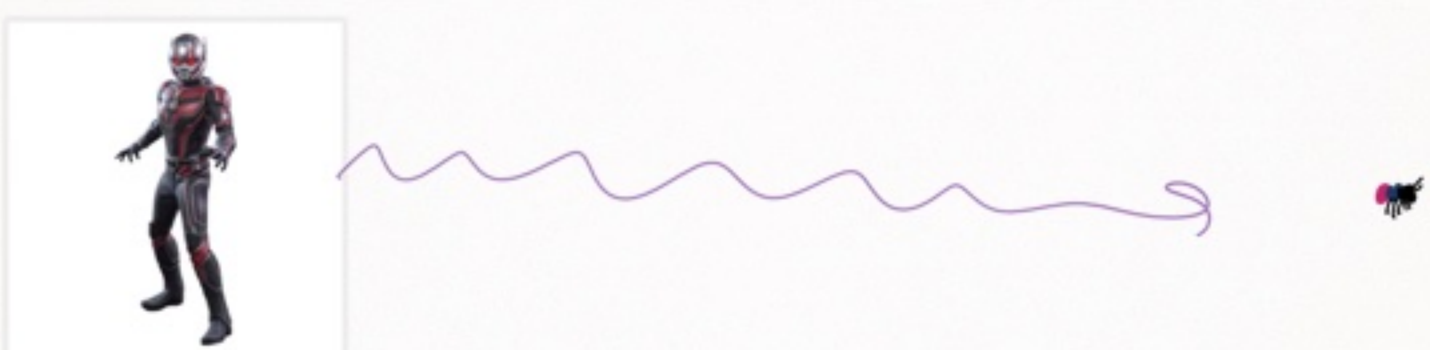
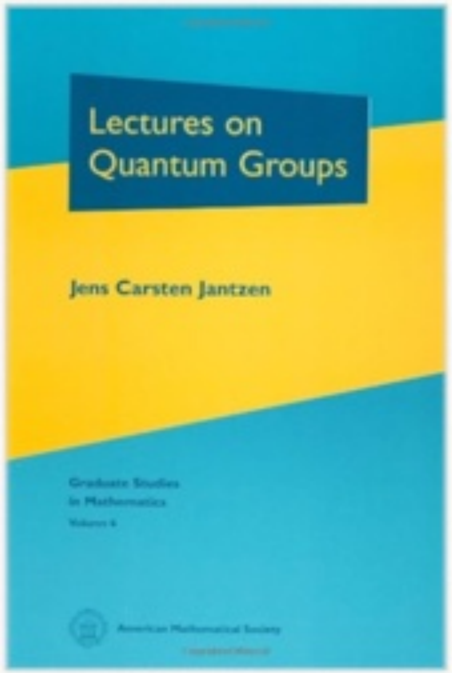
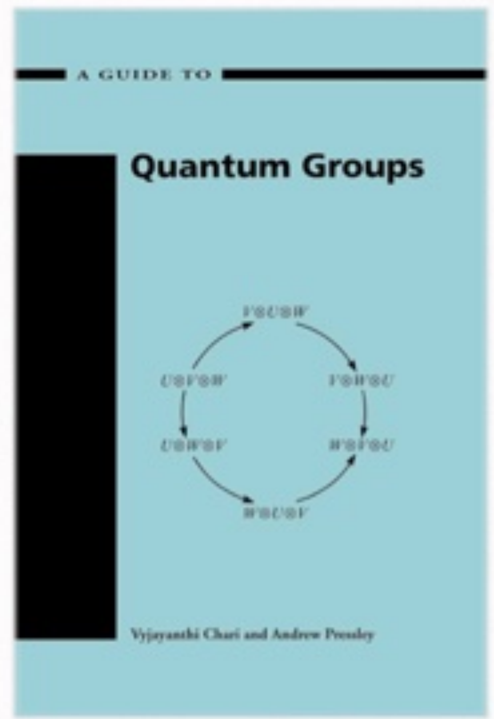
Quantizations!



Extense bibliography, among others

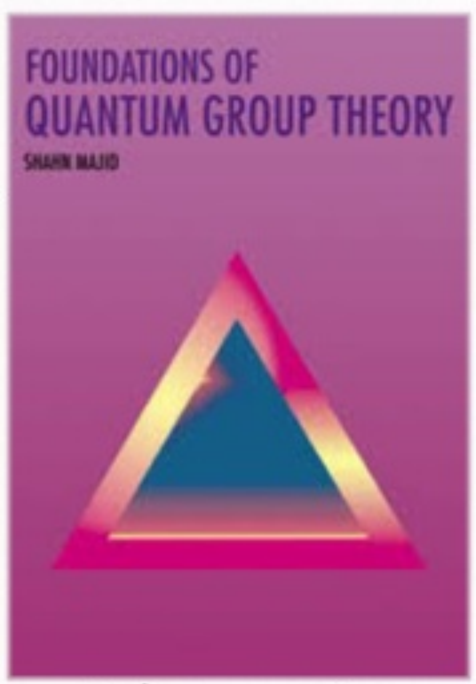


Jantzen



Chari
Presley

Etingof
Schiffman



Majid

Lie algebras

"Continuous Symmetries"

↳ Lie Groups

↓ "Linear" side

Sophus Lie
(1842 - 1899)
Norway

Def A **Lie algebra** is a vector space $\mathfrak{g}$ endowed with a bilinear map

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} \qquad \textbf{bracket} \text{ s.t.}$$

- $[x,y] = -[y,x] \quad \forall x,y \in \mathfrak{g}$ (anti-symmetry)
- $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$ (Jacobi id)

Example

Any associative algebra A is a Lie algebra with bracket

$$[a,b] = ab - ba \qquad \forall a,b \in A$$

For example, $A = M_n(\mathbb{k}) = \mathfrak{gl}_n(\mathbb{k})$

Also, $\mathfrak{sl}_n(\mathbb{k}) = \{A \in M \mid \operatorname{tr} A = 0\}$ is a Lie algebra

Rmk: A Lie algebra is not an associative algebra with the bracket

<u>Idea</u>: Construct the "minimal" assoc. algebra containing $\mathfrak{g}$:

Universal enveloping algebra

Let $\mathfrak{g}$ be a Lie algebra

$T(\mathfrak{g})$ = tensor algebra in $\mathfrak{g}$

<u>Def</u>: $U(\mathfrak{g}) = T(\mathfrak{g}) \big/ \left([x,y] - (x \otimes y - y \otimes x) \right)$

Remarks

1) $U(\mathfrak{g})$ is an associative alg. and a Lie algebra as above

2) $\exists$ Lie alg map $\mathfrak{g} \hookrightarrow U(\mathfrak{g})$ (mono = PBW theorem)

3) $U(\mathfrak{g})$ is a Hopf algebra

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$
$$\varepsilon(x) = 0 \qquad \forall x \in \mathfrak{g}$$
$$S(x) = -x$$

COCOMMUTATIVE

Def A Lie algebra $\mathfrak{g}$ is *simple* if

- it is not abelian

  $([x,y]=0 \ \forall\, x,y\in\mathfrak{g})$

- the only ideals are $\{0\}$ and $\mathfrak{g}$

  $\mathfrak{a}\subseteq\mathfrak{g}, \quad [\mathfrak{g},\mathfrak{a}]\subseteq\mathfrak{a}$

Example $\mathfrak{g}=\mathfrak{sl}_n(\mathbb{C})$ is simple

(Exercise!)

Finite-dimensional (semi) simple Lie algebras over $\mathbb{C}$ are classified!

Cartan matrices and/or Dynkin diagrams

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Def A Cartan matrix of finite type is $A = (a_{ij})_{1 \le i,j \le n}$:

- $a_{ij} \in \mathbb{Z} \ \forall i,j$
- $a_{ii} = 2 \ \forall i$
- $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$
- $a_{ij} < 0$ if $a_{ij} \neq 0$
- $a_{ij} a_{ji} < 4 \ (i \neq j)$

Theorem (Killing-Cartan)

$A = (a_{ij})_{1 \le i,j \le n}$ Cartan matrix

$\Rightarrow \exists$ (semi)simple Lie algebra with generators H_i, E_i, F_i, $1 \le i \le n$ and rels:

$$\left.\begin{aligned} &[H_i, H_j] = 0 \\ &[H_i, E_j] = a_{ij} E_j \qquad \forall i,j \\ &[H_i, F_j] = -a_{ij} F_j \\ &[E_i, F_j] = \delta_{ij} H_i \end{aligned}\right\} \text{Chevalley relations}$$

$$\left.\begin{aligned} &\mathrm{ad}^{(1-a_{ij})}(E_i)(E_j) = 0 \\ &\mathrm{ad}^{(1-a_{ij})}(F_i)(F_j) = 0 \end{aligned}\right\} \text{Serre relations}$$

Conversely, any finite-dimensional (semi) simple Lie algebras is isomorphic to one of these $\mathfrak{g}(A)$.

<u>Example</u> $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, $\dim \mathfrak{g} = 3$

$A = (2)$. <u>Generators</u>: H, E, F

<u>Relations</u>: $[H,E] = 2E$, $[H,F] = -2F$

$[E,F] = H$ (no Serre relations)

Quantization of $U(\mathfrak{g})$

$\hbar$ formal variable

$\Bbbk[[\hbar]]$ = ring of formal power series

<u>Idea</u>: For $\mathfrak{g}$ simple, deform $U(\mathfrak{g})$ over $\Bbbk[[\hbar]]$ s.t

$$U_\hbar(\mathfrak{g}) \xrightarrow[\hbar \to 0]{} U(\mathfrak{g})$$

Definition (Drinfeld-Jimbo $U_\hbar(\mathfrak{g}(A))$)

$A := \left(a_{i,j} \right)_{1 \leq i,j \leq n}$ finite Cartan matrix and $D = \mathrm{diag}(d_1, \dots, d_n)$, $d_i \in \mathbb{Z} - \{0\}$ with DA symmetric. Set $q = e^{\hbar}$, $q_i = e^{\hbar\, d_i}$.

$U_\hbar(\mathfrak{g}(A))$ is the (topological, $\hbar$–adically complete) $\Bbbk[[\hbar]]$-algebra generated by H_i, E_i, F_i for all $1 \leq i \leq n$ with

$$H_i\, E_j - E_j\, H_i = a_{ij}\, E_j \ , \qquad H_i\, F_j - F_j\, H_i = -a_{ij}\, F_j$$

$$H_i\, H_j = H_j\, H_i \ , \qquad E_i\, F_j - F_j\, E_i = \delta_{i,j}\, \frac{e^{+\hbar\, d_i H_i} - e^{-\hbar\, d_i H_i}}{q_i^{+1} - q_i^{-1}}$$

$$\mathrm{ad}_c(E_i)^{1-a_{ij}}(E_j) = \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} E_i^{1-a_{ij}-k} E_j E_i^k = 0 \quad (i \neq j)$$

$$\mathrm{ad}_c(F_i)^{1-a_{ij}}(F_j) = \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} F_i^{1-a_{ij}-k} F_j F_i^k = 0 \quad (i \neq j)$$

Here $$e^{\hbar d_i H_i} = \sum_{n \geq 0} \frac{\hbar^n d_i^n}{n!} H_i^n$$

Note:

$$\frac{e^{\hbar d_i H_i} - e^{-\hbar d_i H_i}}{q_i - q_i^{-1}} = H_i + O(\hbar)$$

Theorem $U_\hbar(\mathfrak{g}(A))$ is a topological Hopf algebra which is neither commutative nor cocommutative

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \varepsilon(H_i) = 0, \quad S(H_i) = -H_i$$

$$\Delta(e^{\hbar d_i H_i}) = e^{\hbar d_i H_i} \otimes e^{\hbar d_i H_i}, \quad \varepsilon(e^{\hbar d_i H_i}) = 1$$

$$S(e^{\hbar d_i H_i}) = e^{-\hbar d_i H_i}$$

$$\Delta(E_i) = E_i \otimes 1 + e^{\hbar d_i H_i} \otimes E_i, \quad \varepsilon(E_i) = 0, \quad S(E_i) = -e^{-\hbar d_i H_i} E_i$$

$$\Delta(F_i) = F_i \otimes e^{-\hbar d_i H_i} + 1 \otimes F_i, \quad \varepsilon(F_i) = 0, \quad S(F_i) = -F_i e^{\hbar d_i H_i}$$

Theorem $U_\hbar(\mathfrak{g}(A)) \cong U(\mathfrak{g}(A))[[\hbar]]$
as algebras (but not as coalgebras)

Theorem (Semi-classical limit)

$$U_\hbar(\mathfrak{g}(A)) \big/ \hbar\, U_\hbar(\mathfrak{g}(A)) \cong U(\mathfrak{g}(A))$$

(i.e. when $\hbar \longrightarrow 0$)

Polynomial version

- $q = e^{\hbar} \in \Bbbk[[\hbar]]$

  $\Rightarrow$ ring $\Bbbk[q, q^{-1}] \subseteq \Bbbk[[\hbar]]$

- $\Bbbk(q)$ field of fractions of $\Bbbk[q, q^{-1}]$

Def $U_q(\mathfrak{g})$ is the $\Bbbk(q)$-algebra generated by $k_i^{\pm 1} = e^{\pm \hbar d_i H_i}$, $E_i, F_i \ \forall i$ satisfying the relations for $U_\hbar(\mathfrak{g})$

Theorem $U_q(\mathfrak{g})$ is a usual Hopf algebra with:

$$k_i k_j = k_j k_i \quad , \quad k_i k_i^{-1} = 1$$

$$k_i E_j k_i^{-1} = q^{d_i a_{ij}} E_j \quad , \quad k_i F_j k_i^{-1} = q^{-d_i a_{ij}} F_j$$

$$[E_i, F_j] = \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}$$

- q-serre relations

$$\Delta(k_i) = k_i \otimes k_i, \quad \varepsilon(k_i) = 1, \quad S(k_i) = k_i^{-1}$$

$$\Delta(E_i) = E_i \otimes 1 + k_i \otimes E_i, \quad \varepsilon(E_i) = 0, \quad S(E_i) = -k_i^{-1} E_i$$

$$\Delta(F_i) = F_i \otimes k_i^{-1} + 1 \otimes F_i, \quad \varepsilon(F_i) = 0, \quad S(F_i) = -F k_i$$

Example $U_q(sl_2)$ generated by k, k^{-1}, E, F with

$$kEk^{-1} = q^2 E \ , \quad kFk^{-1} = q^{-2} F$$

$$EF - FE = \frac{k - k^{-1}}{q - q^{-1}}$$

It is NON-COMMUTATIVE and NON-COCOMMUTATIVE

$$\Delta(k) = k \otimes k \ , \ \varepsilon(k) = 1, \ S(k) = k^{-1}$$

$$\Delta(E) = E \otimes 1 + k \otimes E \ , \ \varepsilon(E) = 0, \ S(E) = -k^{-1}E$$

$$\Delta(F) = F \otimes k^{-1} + 1 \otimes F, \ \varepsilon(F) = 0, \ S(F) = -Fk$$

Borel subalgebras

$U_q(\mathfrak{b}^+)$ = subalg. of $U_q(\mathfrak{g}(A))$ generated by $K_i, K_i^{-1}, E_i, \forall i$

$U_q(\mathfrak{b}^-)$ = subalg of $U_q(\mathfrak{g}(A))$ generated by $K_i, K_i^{-1}, F_i \ \forall i$

Theorem $U_q(\mathfrak{b}^+)$ and $U_q(\mathfrak{b}^-)$ are Hopf subalgebras of $U_q(\mathfrak{g}(A))$.

<u>Example</u> $U_q(\mathfrak{b}^+)$ generated by

k, k^{-1}, E with

$$kEk^{-1} = q^2 E$$

It is NON-COMMUTATIVE and NON-COCOMMUTATIVE

$$\Delta(k) = k \otimes k, \quad \varepsilon(k) = 1, \quad S(k) = k^{-1}$$

$$\Delta(E) = E \otimes 1 + k \otimes E, \quad \varepsilon(E) = 0, \quad S(E) = -k^{-1}E$$

Drinfeld Double

Construction of a Hopf algebra from two dualy paired Hopf algebras.

Def H, K Hopf algebras with S inv.

A **Hopf pairing** is a bilinear map $\langle\, ,\rangle : H \otimes K \longrightarrow \mathbb{k}$ such that

$$\langle h, k k' \rangle = \langle h_{(1)}, k \rangle \langle h_{(2)}, k' \rangle, \quad \langle h, 1 \rangle = \varepsilon(h)$$

$$\langle h h', k \rangle = \langle h, k_{(1)} \rangle \langle h', k_{(2)} \rangle, \quad \langle 1, k \rangle = \varepsilon(k)$$

$$\langle S(h), k \rangle = \langle h, S(k) \rangle$$

Drinfeld Double

Construction of a Hopf algebra from two dualy paired Hopf algebras.

Def H, K Hopf algebras with S inv.

A skew-Hopf pairing is a bilinear map $\langle \, , \rangle : H \otimes K \longrightarrow \mathbb{k}$ such that

$$\langle h, k k' \rangle = \langle h_{(1)}, k \rangle \langle h_{(2)}, k' \rangle \; , \; \langle h, 1 \rangle = \varepsilon(h)$$

$$\langle h h', k \rangle = \langle h, k_{(2)} \rangle \langle h', k_{(1)} \rangle , \; \langle 1, k \rangle = \varepsilon(k)$$

$$\langle S(h), k \rangle = \langle h, S^{-1}(k) \rangle$$

Def H, K with bijective antipode

$\langle\ ,\ \rangle : H \otimes K \to \mathbb{k}$ a skew-Hopf pairing

The Drinfeld double is

$$D(H,K) = T(H \oplus K) / I$$

I = two-sided ideal generated by

$1 = 1_H = 1_K \quad , \quad h \otimes h' = hh' \quad , \quad k \otimes k' = kk'$

$$k_{(1)} \otimes h_{(1)} \langle h_{(2)}, k_{(2)} \rangle = \langle h_{(1)}, k_{(1)} \rangle h_{(2)} \otimes k_{(2)}$$

(commutation relation)

Proposition There exists a skew-Hopf pairing

$$\langle\ ,\ \rangle : U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{b}^-) \longrightarrow \Bbbk(q)$$

$$\langle K_i, K_j \rangle = q_i^{a_{ij}} \qquad \langle E_i, F_j \rangle = \frac{-\delta_{ij}}{q_i - q_i^{-1}}$$

$$\langle E_i, K_j \rangle = 0 = \langle K_i, F_j \rangle$$

Theorem $\exists$ Hopf algebra epim.

$$D(U_q(\mathfrak{b}^+), U_q(\mathfrak{b}^-)) \xrightarrow{\pi} U_q(\mathfrak{g}(A))$$

Relation with $\mathcal{O}_q(G)$

G = Lie group assoc. with $\mathfrak{g}(A)$

<u>Proposition</u>: There exists a perfect Hopf pairing

$$\langle\, ,\rangle : U_q(\mathfrak{g}(A)) \otimes \mathcal{O}_q(G) \longrightarrow \mathbb{k}(q)$$

In particular,

$$U_q(\mathfrak{g}(A)) \subseteq \mathcal{O}_q(G)^\circ$$
$$\mathcal{O}_q(G) \subseteq U_q(\mathfrak{g}(A))^\circ$$

Example

$$\langle\ ,\ \rangle : U_q(\mathfrak{gl}_n) \otimes \mathcal{O}_q(\mathfrak{gl}_n) \longrightarrow k$$

$$\langle K_i, X_{st}\rangle = \delta_{st}\, q^{\delta_{i,s} - \delta_{i+1,t}}$$

$$\langle E_i, X_{st}\rangle = \delta_{is}\,\delta_{i+1,t}$$

$$\langle F_i, X_{st}\rangle = \delta_{i+1,s}\,\delta_{i,t}$$

Specializations at roots of 1

As we did for $\mathcal{O}_q(SL_n(\mathbb{k}))$, one may define $U_q(\mathfrak{g}(A))$ for $q \in \mathbb{k} - \{0, 1, -1\}$

Let $\ell \in \mathbb{N}$ be odd and take

q ℓ-th primitive root of unity

($q^\ell = 1$ and if $q^m = 1 \Rightarrow \ell | m$)

Define $U_q(\mathfrak{g}(A))$ as above

The results still hold!!

Proposition

(a) The elements $K_i^\ell, E_i^\ell, F_i^\ell$ are central in $U_q(\mathfrak{g}(A))$

(b) $\Delta(K_i^\ell) = K_i^\ell \otimes K_i^\ell$

$\Delta(E_i^\ell) = E_i^\ell \otimes 1 + K_i^\ell \otimes E_i^\ell$

$\Delta(F_i^\ell) = F_i^\ell \otimes K_i^{-\ell} + 1 \otimes F_i^\ell$

(c) The subalgebra Z_0 of $U_q(\mathfrak{g}(A))$ generated by $K_i^\ell, E_i^\ell, F_i^\ell$ $\forall i$ is a central Hopf subalgebra

Def The Frobenius-Lusztig kernel or small quantum group is

$$u_q(\mathfrak{g}(A)) = U_q(\mathfrak{g}(A)) \big/ (E_i^\ell, F_i^\ell, K_i^\ell - 1)$$

Analogously, $u_q(\mathfrak{b}^+) = U_q(\mathfrak{b}^+) \big/ (E_i^\ell, K_i^\ell - 1)$

$$u_q(\mathfrak{b}^-) = U_q(\mathfrak{b}^-) \big/ (F_i^\ell, K_i^\ell - 1)$$

Example $u_q(sl_2)$

Generators: K, E, F

Relations:

$$K^\ell = 1 \quad (K^{-1} = K^{\ell-1}), E^\ell = 0, \quad F^\ell = 0$$

$$KEK^{-1} = q^2 E \quad ; \quad KFK^{-1} = q^{-2} F$$

$$\dim u_q(sl_2) = \ell^3$$

$$u_q(b^+) = \Bbbk \langle K, E \mid K^\ell = 1, E^\ell = 0, KEK^{-1} = q^2 E \rangle$$

Theorem Write $\mathfrak{g} = \mathfrak{g}(A)$

1) $u_q(\mathfrak{g})$, $u_q(\mathfrak{b}^{\pm})$ are finite-dimensional Hopf algebras which are NON-COMMUTATIVE and NON-COCOM

2) $\dim u_q(\mathfrak{g}) = \ell^{\dim \mathfrak{g}}$

Example $\mathfrak{g} = \mathfrak{sl}_2$

$$u_q(\mathfrak{b}^+) = \Bbbk\langle k, E \mid k^\ell = 1, E^\ell = 0, kEk^{-1} = q^2 E\rangle$$

$$\Delta(k) = k \otimes k, \qquad \Delta(E) = E \otimes 1 + k \otimes E$$

$$\varepsilon(k) = 1 \qquad \varepsilon(E) = 0$$

$$S(k) = k^{\ell-1} \qquad S(E) = -k^{-1}E$$

$\Rightarrow$ $u_q(\mathfrak{b}^+) = T_{q^2}$ Taft algebra

$\dim T_{q^2} = \ell^2$

Earl Taft
1931 - 2021

Remark We saw in lecture 3 that these exists s.e.s.

$$\mathcal{O}(G) \hookrightarrow \mathcal{O}_q(G) \twoheadrightarrow A$$

with A fin. dim. Hopf algebra

Thm $A \cong \mathcal{U}_q(\mathfrak{g})^*$, so $\exists$ s.e.s

$$\mathcal{O}(G) \hookrightarrow \mathcal{O}_q(G) \twoheadrightarrow \mathcal{U}_q(\mathfrak{g})^*$$

proof Use the perfect pairing between $\mathcal{O}_q(G)$ and $U_q(\mathfrak{g})$ $\blacksquare$