Around proper actions on homogeneous spaces

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(and of convex cocpt right-angled reflection groups in $O(n,1)$)

(Lecture by Maciej Bocheński)

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§ 1 Introduction

Def^n : L : a loc cpt gp. X : a loc cpt sp.
An L-action on X is <u>proper</u>
$\overset{\text{def.}}{\Longleftrightarrow}$ $\forall\ C \subset X$: a cpt subset,
$\{ l \in L \mid l \cdot C \cap C \neq \phi \}$ is a cpt subset of L.

Fact : Γ : a discrete gp. X : a mfd.
$\Gamma \curvearrowright X$ proper. free
$\Rightarrow$ $\Gamma \backslash X$ has a (necessarily unique) mfd str.
which makes $\pi : X \to \Gamma \backslash X$ a covering map.

('$\Leftarrow$' if $\Gamma \curvearrowright X$ is effective)

Rmk : Freeness is not a serious issue :
- Γ : torsion-free, $\Gamma \curvearrowright X$: proper $\Rightarrow$ $\Gamma \curvearrowright X$: free.
- proper. not free $\rightsquigarrow$ $\Gamma \backslash X$: an orbifold.

(not proper $\rightsquigarrow$ $\Gamma \backslash X$: non-Hausdorff, pathological)

G/H : a homogeneous space. $\Gamma \subset G$: a discrete subgp
If $\Gamma \curvearrowright G/H$ is proper and free,
we obtain a manifold $\Gamma \backslash G/H$ locally modelled on G/H.
Such $\Gamma \backslash G/H$ is called a <u>Clifford-Klein form</u>.
(Compact CK forms are often called <u>compact quotients</u> of G/H).

Ex : <u>$\mathcal{H}^{p,q}$</u> $:= \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_{p+q+1} \end{bmatrix} \in \mathbb{RP}^{p+q} \;\middle|\; \sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{q+1} x_{p+j}^2 < 0 \right\}$

is called the <u>pseudo-hyperbolic space</u> of signature (p,q).
($\mathcal{H}^{p,0} = \mathcal{H}^p$. $\mathcal{H}^{0,q} = \mathbb{RP}^q$).

$$O(p, q+1) := \left\{ g \in M(p+q, \mathbb{R}) \;\middle|\; {}^t g \cdot \begin{pmatrix} I_p & \\ & -I_{q+1} \end{pmatrix} \cdot g = \begin{pmatrix} I_p & \\ & -I_{q+1} \end{pmatrix} \right\}$$

$PO(p,q+1) := O(p,q+1)/\{\pm 1\}$

$\curvearrowright \quad O(p,q+1)/p(O(p,q)\times O(1)) \xrightarrow{\cong} \mathbb{H}^{p,q}$ diffeo.

$\hat{\mathbb{H}}^{p,q}$: the universal cover of $\mathbb{H}^{p,q}$

(for $q \geq 2$, $\hat{\mathbb{H}}^{p,q} = O(p,q+1)/O(p,q)$)

Then, (Clifford-Klein forms of $\hat{\mathbb{H}}^{p,q}$)

$\overset{1:1}{\longleftrightarrow}$ (Complete pseudo-Riem. mfd of signature (p,q) with sectional curvature $\equiv -1$.)

If H is cpt, every discrete subgp of G acts properly on G/H.

(G/H has a G-inv. Riem. metric)

__This mini-course__ : The case where H is non-cpt.

(Mostly, G : a non-cpt reductive Lie gp
H : a non-cpt red. subgp of G)

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§ 2 $\prec$, $\sim$, and $\pitchfork$

__Notation__ (Kobayashi '96, with a minor change) :

G : a loc. cpt gp, $H, L \subset G$: closed subsets

(1) $H \prec L \overset{\text{def.}}{\Longleftrightarrow} \exists C \subset G$: cpt subset, $H \subset CLC^{-1}$

('subset modulo cpt')

(2) $H \sim L \overset{\text{def.}}{\Longleftrightarrow} H \prec L$ and $L \prec H$.

('equal modulo cpt')

(3) $H \pitchfork L \overset{\text{def.}}{\Longleftrightarrow} \forall C \subset G$: cpt subset, $CHC^{-1} \cap L$ is cpt.

('proper')

__Rmk__ : (1) If $G' \subset G$ is a closed subgp s.t. $H, L \subset G'$, then $H \prec L$ in G $\underset{\not\Rightarrow}{\Leftarrow}$ $H \prec L$ in G',

$H \pitchfork L$ in G $\underset{\not\Leftarrow}{\Rightarrow}$ $H \pitchfork L$ in G'.

(2) $H \subset G$: a closed subset. $C \subset G$: a cpt subset

$\rightarrow$ CHC^{-1} is closed in G.

*

What is the meaning of $\pitchfork$?

[**Lem.** : G : a loc cpt gp, $H, L \subset G$: closed subgps.

Then $H \pitchfork L$ $\Leftrightarrow$ $L \curvearrowright G/H$: proper.]

Pf : Using the local cptness of G, one can easily see that :

[$\forall\, C' \subset G/H$: a cpt subset, $\exists\, C \subset G$: a cpt subset

s.t. $C' = \pi(C)$, where $\pi : G \longrightarrow G/H$ proj.]

We thus see that :

$L \curvearrowright G/H$: proper

$\Leftrightarrow$ $\forall\, C \subset G$: cpt. $\{\ell \in L \mid \ell \cdot \pi(C) \cap \pi(C) \neq \phi\}$ is cpt

$\{\ell \in L \mid \ell C \cap CH \neq \phi\}$ $=$ $CHC^{-1} \cap L$.

$\Leftrightarrow$ $H \pitchfork L$ in G. ☒

[**Lem.** : G : loc cpt gp. $H, L \subset G$: closed subsets

(1) $H \pitchfork L \Leftrightarrow L \pitchfork H$

(2) If $H' \prec H$, then $H \pitchfork L \Rightarrow H' \pitchfork L$.

(resp. $H' \sim H$) (resp. $\Leftrightarrow$)]

Pf : (1) $\forall\, C \subset G$: cpt,

$H \cap \underbrace{CLC^{-1}}_{\text{closed}} \subset C \cdot (\underbrace{C^{-1}HC \cap L}_{\text{cpt}}) \cdot C^{-1}$ is cpt. OK

(2) Take $C \subset G$: cpt so that $H' \subset CHC^{-1}$.

Then, $\forall C' \subset G$: cpt,

$$\underbrace{C' H' C'^{-1}}_{\text{closed}} \cap L \subset \underbrace{(C'C) H (C'C)^{-1}}_{\text{cpt}} \cap L \quad \text{is cpt.} \quad \text{OK.}$$

[**Cor.** : G : loc cpt gp. $H, L \subset G$: closed subgps

Then, $L \curvearrowright G/H$: proper $\Leftrightarrow$ $H \curvearrowright G/L$: proper.]

Pf : $L \curvearrowright G/H$: proper $\Leftrightarrow$ $H \pitchfork L$

$\Leftrightarrow$ $L \pitchfork H$ $\Leftrightarrow$ $H \curvearrowright G/L$: proper. ▨

[**Th^m** (Calabi - Markus '62, Wolf '62, Kobayashi '89).

$\underline{G/H \text{ : a reductive homogeneous space, } \mathrm{rk}_{\mathbb{R}} G = \mathrm{rk}_{\mathbb{R}} H}$

(will explain in § 3. e.g. $H^{p,q}$ with $p \le q$,

$GL(p+q, \mathbb{R}) / (GL(p, \mathbb{R}) \times GL(q, \mathbb{R}))$)

Then, a closed (resp. discrete) subgp of G acts properly on G/H

if and only if it is cpt (resp. finite).]

Pf for $GL(p+q, \mathbb{R}) / (GL(p, \mathbb{R}) \times GL(q, \mathbb{R}))$:

$K := O(p+q) \subset G$. ($\rightsquigarrow$ $K \cap H = O(p) \times O(q)$).

$$A := \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{p+q} \end{pmatrix} \middle| \ a_1, \ldots, a_{p+q} > 0 \right\}$$

[Singular value decomposition : $G = K \cdot A \cdot K$.]

(proved easily by the Gram - Schmidt process)

We also have $H = (K \cap H) \cdot A \cdot (K \cap H)$.

$\rightsquigarrow$ $G \sim A \sim H$.

Now, for a closed subgp L of G,

$L \curvearrowright G/H$: proper $\Leftrightarrow$ $H \pitchfork L$ $\Leftrightarrow$ $G \pitchfork L$ $\Leftrightarrow$ L is cpt. ▨

§ 3 Reductive Lie groups

(… groups that behave like $GL(n, \mathbb{R})$)

Put $\underline{\theta_{std}}$: $GL(n, \mathbb{R}) \xrightarrow{\sim} GL(n, \mathbb{R})$

$$g \longmapsto {}^t g^{-1}$$

and call it the standard Cartan involution on $GL(n, \mathbb{R})$.

It induces the Lie algebra involution $\theta_{std} : \mathfrak{gl}(n, \mathbb{R}) \xrightarrow{\sim} \mathfrak{gl}(n, \mathbb{R})$.

$$X \longmapsto -{}^t X$$

Def^n : G : a Lie gp. $\theta : G \to G$ an involution

(G, θ) : a reductive Lie gp

$\overset{\text{def.}}{\Longleftrightarrow}$ $\exists\, i : G \hookrightarrow GL(n, \mathbb{R})$ an emb. of Lie gps

$\Big\{$ · $i \circ \theta = \theta_{std} \circ i$

· $i(G)$ is Euclidean-open in its Zariski-closure.

θ is called the Cartan involution on G.

We often omit θ and say that G is a red. Lie gp.

Examples : (1) $GL(n, K)$, $SL(n, K)$ for $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

(For $K = \mathbb{C}, \mathbb{H}$, use $\mathbb{C} \hookrightarrow M(2, \mathbb{R})$, $\mathbb{H} \hookrightarrow M(2, \mathbb{C})$

$a + bi \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $a + bj \mapsto \begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix}$)

(2) $O(p, q) := \{ g \in M(p+q, \mathbb{R}) \mid {}^t g \cdot I_{p,q} \cdot g = I_{p,q} \}$.

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \in M(p+q, \mathbb{R})$$

$\cup$ index 2 if $(p, q) \neq (0, 0)$.

$SO(p, q) := O(p, q) \cap SL(p+q, \mathbb{R})$

$\cup$ index 2 if $p, q \geq 1$

$SO_0(p, q)$:= the identity component of $SO(p, q)$.

(3) Other classical groups :

	symmetric	skew-symmetric	Hermitian	skew-Hermitian
$\mathbb{R}$	$O(p,q)$	$Sp(2n,\mathbb{R})$	$O(p,q)$	$Sp(2n,\mathbb{R})$
$\mathbb{C}$	$O(n,\mathbb{C})$	$Sp(2n,\mathbb{C})$	$U(p,q)$	$U(p,q)$
$\mathbb{H}$	$\times$	$\times$	$Sp(p,q)$	$O^*(2n)$

(4) Exceptional Lie groups.

(5) Products, projectivizations, etc. of the above examples.

e.g. $PO(p,q) := O(p,q) / \{\pm I_n\}$

Notation : $G = (G, \theta)$: a reductive Lie gp.

$K := G^\theta = \{ g \in G \mid \theta(g) = g \}$

$\mathfrak{k} := \mathfrak{g}^\theta = \{ X \in \mathfrak{g} \mid \theta(X) = X \}$

$\mathfrak{p} := \mathfrak{g}^{-\theta} = \{ \text{---}''\text{---} \mid \theta(X) = -X \}$

$\rightsquigarrow$ $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: K-inv. decomp.

$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $\quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $\quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.

Ex : (1) $G = GL(n,\mathbb{R})$

$\rightsquigarrow$ $K = O(n) = \{ g \in GL(n,\mathbb{R}) \mid {}^t g \cdot g = I_n \}$ orthogonal

$\mathfrak{k} = \mathfrak{o}(n) = \{ X \in \mathfrak{gl}(n,\mathbb{R}) \mid {}^t X + X = 0 \}$ skew-symmetric

$\mathfrak{p} = \mathrm{Sym}(n,\mathbb{R}) = \{ \text{---}''\text{---} \mid {}^t X = X \}$ symmetric

(2) $G = O(p,q)$

$\rightsquigarrow$ $\mathfrak{g} = \mathfrak{o}(p,q) = \{ X \in M(p+q,\mathbb{R}) \mid {}^t X \cdot I_{p,q} + I_{p,q} \cdot X = 0 \}$

$K = O(p) \times O(q)$, $\quad \mathfrak{k} = \mathfrak{o}(p) \oplus \mathfrak{o}(q)$.

$$\mathfrak{p} = \left\{ \left(\begin{array}{c|c} 0 & X \\ \hline {}^t X & 0 \end{array} \right) \;\middle|\; X \in M(p,q,\mathbb{R}) \right\}$$

[**Fact A**: $\mathfrak{p} \times K \xrightarrow{\cong} G$ is a diffeo.
$(X, k) \longmapsto \exp(X)\cdot k$]

Skip

We fix a G-inv. symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$
s.t. $\mathfrak{k} \perp \mathfrak{p}$ w.r.t. B. $B|_{\mathfrak{k}\times\mathfrak{k}}$: neg-def. $B|_{\mathfrak{p}\times\mathfrak{p}}$: pos-def.

(Such B always exists : regard $G \subset GL(n, \mathbb{R})$ and put
$B(X, Y) := \mathrm{tr}(XY)$)

Fix a maximal abelian subsp. $\mathfrak{a}$ of $\mathfrak{p}$, and put $A := \exp(\mathfrak{a})$

($\mathfrak{a} \xrightarrow{\exp} A$ is a diffeo. by Fact A)

Ex: (1) $G = GL(n, \mathbb{R})$

$$\mathfrak{a} = \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \;\middle|\; t_1, \dots, t_n \in \mathbb{R} \right\}, \quad A = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \;\middle|\; a_1, \dots, a_n > 0 \right\}$$

(2) $G = O(p, q)$ with $p \geq q$.

<u>Work with another model</u> :

$O(p, q) := \{ g \in M(p+q, \mathbb{R}) \mid {}^t g \cdot I'_{p,q} \cdot g = I'_{p,q} \}$, where

$$I'_{p,q} := \left(\begin{array}{c|c|c} 0 & 0 & S_q \\ \hline 0 & I_{p-q} & 0 \\ \hline S_q & 0 & 0 \end{array}\right) \in M(p+q, \mathbb{R}). \quad S_q := \begin{pmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{pmatrix} \in M(q, \mathbb{R})$$

Then,
$$\mathfrak{a} = \left\{ \left(\begin{array}{c|c|c} \begin{matrix} t_1 & & \\ & \ddots & \\ & & t_q \end{matrix} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & \begin{matrix} -t_q & & \\ & \ddots & \\ & & -t_1 \end{matrix} \end{array}\right) \;\middle|\; t_1, \dots, t_q \in \mathbb{R} \right\}$$

$$A = \left\{ \left(\begin{array}{c|c|c} \begin{matrix} a_1 & & \\ & \ddots & \\ & & a_q \end{matrix} & 0 & 0 \\ \hline 0 & I_{p-q} & 0 \\ \hline 0 & 0 & \begin{matrix} a_q^{-1} & & \\ & \ddots & \\ & & a_1^{-1} \end{matrix} \end{array}\right) \;\middle|\; a_1, \dots, a_q > 0 \right\}$$

Fact B : (1) $K \times \mathfrak{a} \twoheadrightarrow \mathfrak{p}$ is surj.

$(k, X) \longmapsto \mathrm{Ad}(k) X$

(2) $K \times A \times K \twoheadrightarrow G$ is surj.

$(k_1, a, k_2) \longmapsto k_1 \cdot a \cdot k_2$

(3) Up to conj. by G, every compact subgp of G is contained in K.

(4) ———— " ———— K, every abelian subsp of $\mathfrak{p}$ ———— " ———— $\mathfrak{a}$.

(Easy : Fact A + (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (4))

Def^n : Put $\mathrm{rk}_{\mathbb{R}}\, G := \dim \mathfrak{a}$ and call it the $\mathbb{R}$-rank of G.
(well-def. by Fact B (4))

Ex. : (1) $\mathrm{rk}_{\mathbb{R}}\, GL(n, \mathbb{R}) = n$.

(2) $\mathrm{rk}_{\mathbb{R}}\, O(p, q) = \min\{p, q\}$.

Def^n : We define the restricted Weyl group $W = W(\mathfrak{g}; \mathfrak{a})$ by
$W := N_K(\mathfrak{a}) / Z_K(\mathfrak{a})$.

W is a finite group and acts faithfully on $\mathfrak{a}$.

Ex. : (1) $G = GL(n, \mathbb{R}) \rightsquigarrow W \cong \mathfrak{S}_n$,
$W \curvearrowright \mathfrak{a} \cong \mathbb{R}^n$ by coordinate permutations

(2) $G = O(p, q)$ with $p \geq q \rightsquigarrow W \cong \mathfrak{S}_q \ltimes \{\pm 1\}^q$
$W \curvearrowright \mathfrak{a} \cong \mathbb{R}^q$ by coord. permutations / sign changes.

* **Rmk** : G_0 : the identity component of G.
$K_0 := K \cap G_0$. $W_0 := N_{K_0}(\mathfrak{a}) / Z_{K_0}(\mathfrak{a})$.
$\rightsquigarrow$ induces an embedding $W_0 \hookrightarrow W$.
W_0 is canonically isom. to the 'root-theoretic' Weyl group.

It is common to assume that G satisfies $W_0 = W$.

We will not do that until it is really needed (in § 5)

Non-ex. : $G = O(n,n)$ (or $PO(n,n)$) $(n \geq 1)$

$$\begin{array}{lll} G = O(n,n), & W = \mathfrak{S}_n \ltimes \{\pm 1\}^n & \\ \cup & \cup \text{ index } 2 & \\ G_0 = SO_0(n,n), & W_0 = \{ \sigma \ltimes (\varepsilon_i)_{1 \leq i \leq n} \in W \mid \prod_{i=1}^{n} \varepsilon_i = 1 \} & \end{array}$$

Fact C : For $X, X' \in \mathfrak{a}$, the following (i) - (iii) are equivalent :

(i) $W \cdot X = W \cdot X'$ in $\mathfrak{a}$

(ii) $\mathrm{Ad}(K) \cdot X = \mathrm{Ad}(K) \cdot X'$ in $\mathfrak{p}$.

(iii) $K \cdot \exp(X) \cdot K = K \cdot \exp(X') \cdot K$ in G.

(Easy : (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii))

Def^n : We define the Cartan projection $\mu : G \longrightarrow \mathfrak{a}/W$ by

$\mu(\underset{K}{\underset{\cap}{k_1}} \exp(\underset{\mathfrak{a}}{\underset{\cap}{X}}) \underset{K}{\underset{\cap}{k_2}}) := W \cdot X$. (well-def. by Fact B (2) and Fact C)

Def^n : $G = (G, \theta)$: a red. Lie gp.

A closed subgp H of G is called a reductive subgp if

$\theta(H) = H$ and $(H, \theta|_H)$ is a red. Lie gp.

⇝ $K_H := K \cap H$, $\mathfrak{k}_H := \mathfrak{k} \cap \mathfrak{h}$, $\mathfrak{p}_H := \mathfrak{p} \cap \mathfrak{h}$.

A homog. sp. G/H is called reductive (or of reductive type)

if G is red. and H is a red. subgp of G.

If $H \subset G$ is reductive, up to conj. by G,

$\mathfrak{a}_H := \mathfrak{a} \cap \mathfrak{h}$ is a max. abelian subsp. of $\mathfrak{p}_H := \mathfrak{p} \cap \mathfrak{h}$.

We always assume that this is satisfied.

In this situation, $H = K_H \cdot \exp(\mathfrak{a}_H) \cdot K_H$

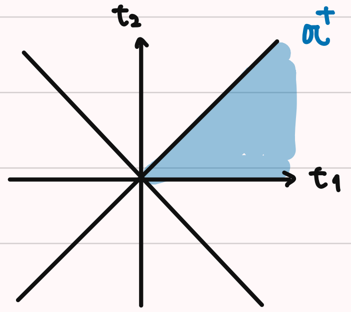
$\rightsquigarrow \mu(H) = W \cdot \mathfrak{a}_H \subset \mathfrak{a}/W$.

<u>Ex</u> : $G = O(p, q)$ with $p \geq q \geq 1$.

$\rightsquigarrow \mathfrak{a} = \mathbb{R}^q$. $W = \mathfrak{S}_q \ltimes \{\pm 1\}^q$.

We identify $\mathfrak{a}/W$ with $\mathfrak{a}^+ := \left\{ \begin{pmatrix} t_1 \\ \vdots \\ t_q \end{pmatrix} \middle| \; t_1 \geq \cdots \geq t_q \geq 0 \right\} \subset \mathfrak{a}$.

($\underline{q=2}$:

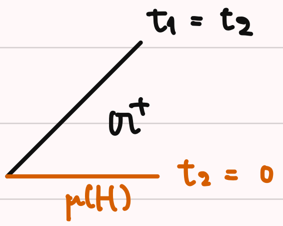


)

(1) $H := O(p, q-1) \subset G \rightsquigarrow \mathfrak{a}_H = \mathbb{R}^{q-1} \subset \mathbb{R}^q = \mathfrak{a}$.

We have $\mu(H) = \left\{ \begin{pmatrix} t_1 \\ \vdots \\ t_{q-1} \\ 0 \end{pmatrix} \middle| \; t_1 \geq \cdots \geq t_{q-1} \geq 0 \right\} \subset \mathfrak{a}^+$.

($\underline{q=2}$:



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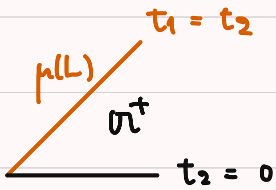
(2) Assume $p = 2p'$ and $q = 2q'$ are even.

$L := U(p', q') \subset O(2p', 2q') = G$.

$\rightsquigarrow \mathfrak{a}_L = \left\{ \begin{pmatrix} t_1 \\ t_1 \\ \vdots \\ t_{q'} \\ t_{q'} \end{pmatrix} \middle| \; t_1, \ldots, t_{q'} \in \mathbb{R} \right\} \subset \mathbb{R}^{2q'} = \mathfrak{a}$.

We have $\mu(L) = \left\{ \begin{pmatrix} t_1 \\ t_1 \\ \vdots \\ t_{q'} \\ t_{q'} \end{pmatrix} \middle| \; t_1 \geq \cdots \geq t_{q'} \geq 0 \right\} \subset \mathfrak{a}^+$.

($q = 2$:



)

Skip

Geometric interpretation of μ :

G/K : the Riemannian symmetric space. $p_0 := 1 \cdot K \in G/K$

$\mathrm{Exp}_{p_0} : \mathfrak{p} \xrightarrow{\simeq} G/K$ is a diffeo.

$$\begin{array}{ccc} \mathfrak{p} & \xrightarrow{\simeq} & G/K \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ X & \longmapsto & \exp(X) \cdot p_0 \end{array}$$

$B|_{\mathfrak{p}\times\mathfrak{p}}$ induces a G-inv. Riem. metric of non-positive sectional curvature on G/K. We write $d = d_{G/K}$ for the associated distance.

[**Lem** : $\forall g \in G$, $\| \mu(g) \| = d_{G/K}(g \cdot p_0, p_0)$.]

(Idea : If $\mathrm{rk}_{\mathbb{R}} G = 1$, μ is just the distance from p_0.
If $\mathrm{rk}_{\mathbb{R}} G > 1$, μ is an 'enhancement' of the distance !)

Pf of Lem : For $g = k_1 \exp(X) k_2$ ($k_1, k_2 \in K$, $X \in \mathfrak{a}$), we have

$$d_{G/K}(g \cdot p_0, p_0) = d_{G/K}(k_1 \exp(X) \underbrace{k_2 \cdot p_0}_{= p_0}, p_0) \underset{(d_{G/K} :\ G\text{-inv.})}{=} d_{G/K}(\exp(X) p_0, \underbrace{k_1^{-1} p_0}_{= p_0})$$

$$= d_{G/K}(\exp(X) p_0, p_0) = \|X\|.$$

∎

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§ 4 The properness criterion and its applications

Thm (Kobayashi '89, '96, Benoist '96) :

G : a red Lie gp. $H, L \subset G$: a closed subset.

Then, $H \pitchfork L$ in G $\Leftrightarrow$ $\mu(H) \pitchfork \mu(L)$ in $\mathfrak{a}$

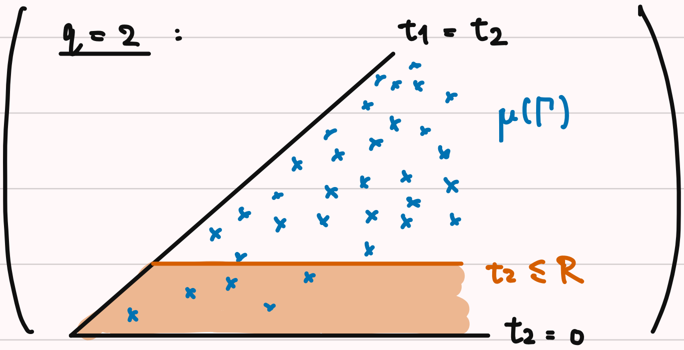
regard as W-inv. subsets of $\mathfrak{a}$

$\Big(\Leftrightarrow \ {}^\forall R \geq 0.\ \underset{\uparrow \text{(closed } R\text{-nbd)}}{N_R}(\mu(H)) \cap \mu(L)$ is compact in $\mathfrak{a}/W.\Big)$

Ex : $G/H = \mathcal{H}^{p,q-1} = PO(p,q)/P(O(p,q-1)\times O(1))$ $(p \geq q \geq 1)$.

Identify $\mathfrak{a}/W$ with $\mathfrak{a}^+ = \left\{ \begin{pmatrix} t_1 \\ \vdots \\ t_q \end{pmatrix} \;\middle|\; t_1 \geq \cdots \geq t_q \geq 0 \right\}$

A discrete subgp Γ of G acts properly on $\mathcal{H}^{p,q-1}$ if and only if, ${}^\forall R \geq 0$, $\mu(\Gamma) \cap \{t_q \leq R\}$ is a finite subset of $\mathfrak{a}^+$.



If both H and L are reductive subgps of G, the properness criterion has a simpler form :

(in Kobayashi '89)

Properness criterion, the reductive case :

G : a red Lie gp. $H, L \subset G$: red. subgps.

(WLOG $\mathfrak{a}_H, \mathfrak{a}_L \subset G$).

Then, $L \curvearrowright G/H$: proper $\Leftrightarrow$ $W \cdot \mathfrak{a}_H \cap W \cdot \mathfrak{a}_L = \{0\}$

Def^n : G/H : red. Γ : a discrete subgp of G

$\Gamma \curvearrowright G/H$ is a standard proper action

$\Leftrightarrow$ $\Gamma \subset {}^\exists L \underset{\text{red.}}{\subset} G$ s.t. $L \curvearrowright G/H$: proper.

A proper action is called exotic if it is not a deformation of any standard proper action.

Let us give a first application of the criterion :

$\underline{\text{Th}^m}$ (Kobayashi) :

For G/H : red, the following (i) ~ (iii) are equivalent :

(i) $\exists\, \Gamma$: a discrete subgp of G acting properly $\curvearrowright G/H$ and isom. to $\mathbb{Z}$.

(ii) $\exists\, L$: a red. subgp of G acting properly $\curvearrowright G/H$ and isom. to $\mathbb{R}$.

(iii) $\mathrm{rk}_{\mathbb{R}}\, G > \mathrm{rk}_{\mathbb{R}}\, H$

($\exists$ a proper $\mathbb{Z}$ - action $\Leftrightarrow$ $\exists$ a standard proper $\mathbb{Z}$ - action)

$\underline{\text{Pf}}$: $\underline{\text{(i)} \Rightarrow \text{(iii)}}$: Proved already in § 2.

$\underline{\text{(ii)} \Rightarrow \text{(i)}}$: Take Γ to be a copy of $\mathbb{Z}$ in $L \cong \mathbb{R}$.

$\underline{\text{(iii)} \Rightarrow \text{(ii)}}$: $\mathrm{rk}_{\mathbb{R}}\, G > \mathrm{rk}_{\mathbb{R}}\, H \leadsto \mathfrak{a}_H \subsetneq \mathfrak{a}$,

Choose a 1 - dim^l subsp. $\ell \subset \mathfrak{a}$ so that $\underline{\ell \cap W \cdot \mathfrak{a}_H = \{0\}}$,

$\Updownarrow$

and put $L := \exp(\ell) \subset G$. $\quad W \cdot \mathfrak{a}_H \cap W \cdot \ell = \{0\}$

Then $L \curvearrowright G/H$ is proper by the criterion. ∎

$\underline{\text{Proper cocpt actions}}$:

(Usually $\mathbb{Z}$ or $\mathbb{R}$)

$\underline{\text{Def}^n}$: R : a ring, Γ : a virtually torsion - free gp G,

Take $\Gamma_0 \subset \Gamma$: finite - index, torsion - free and put

$$\underline{\mathrm{vcd}_R(\Gamma)} := \sup \left\{ p \in \mathbb{N} \;\middle|\; \begin{array}{l} H^p(\Gamma_0 ; E) \neq 0 \text{ for} \\ \text{some } R\Gamma_0\text{-module } E \end{array} \right\} \in \mathbb{N} \cup \{\infty\}$$

(It does not depend on the choice of Γ_0).

It is called the R - virtual cohomological dimension of Γ.

If $\Gamma \subset G$ for a $\underline{\text{reductive}}$ Lie gp G, then

↑ not needed

$$\mathrm{vcd}_R(\Gamma) := \sup \left\{ p \in \mathbb{N} \;\middle|\; H^p(\Gamma_0 \backslash G/K ; \mathcal{E}) \text{ for some } R\text{-local system } \mathcal{E} \right\}$$

$\leadsto \mathrm{vcd}_R(\Gamma) \leq \dim \mathfrak{p}$, with equality precisely when $\Gamma \subset G$: cocpt.

Fact (Bestvina - Mess) :

Γ : a virtually torsion-free Gromov-hyperbolic gp.

$\dim \partial\Gamma :=$ the covering dimension of Γ.

If $\dim \partial\Gamma < \infty$, then $\mathrm{vcd}_{\mathbb{Z}}\, \Gamma = \dim \partial\Gamma + 1$.

for $\mathbb{H}^{p,q}$

Thm (Kulkarni, Kobayashi) : R : a ring,

G/H : red, $\Gamma \subset G$: a discrete subgp.

not needed

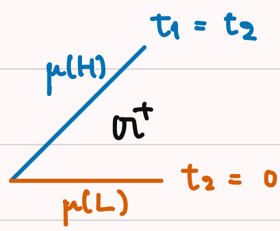
$\Gamma \curvearrowright G/H$: proper $\Rightarrow$ $\mathrm{vcd}_R(\Gamma) \leq \dim \mathfrak{p} - \dim \mathfrak{p}_H$

with equality precisely when $\Gamma \curvearrowright G/H$ is cocpt.

Let us give an example of (standard) proper cocpt actions :

Ex : $G = O(2m, 2)$ $(m \geq 1)$.

$H = U(m, 1)$. $L = O(2m, 1)$ $\rightsquigarrow$ $H \pitchfork L$



(1) Take a cocpt lattice Γ_L of L.

$\mathrm{vcd}\, \Gamma_L = \dim \mathfrak{p}_L = 2m = \dim \mathfrak{p} - \dim \mathfrak{p}_H$

$\rightsquigarrow$ $\Gamma_L \curvearrowright G/H$: proper. cocpt.

(2) Take a cocpt lattice Γ_H of H.

$\mathrm{vcd}\, \Gamma_H = \dim \mathfrak{p}_H = 2m = \dim \mathfrak{p} - \dim \mathfrak{p}_L$

$\rightsquigarrow$ $\Gamma_H \curvearrowright G/L$: proper, cocpt.

The following is one of the biggest open problems in this area :

Conj (Kobayashi) :

For G/H : red, the following (i) and (ii) are equivalent :

(i) $\exists\, \Gamma \subset G$: a discrete subgp s.t. $\Gamma \curvearrowright G/H$: proper cocpt.

(ii) $\exists\, L \subset G$: a reductive subgp s.t. $L \curvearrowright G/H$: —〃— .

Rmk : (1) (ii) $\Rightarrow$ (i) is obvious (Γ := a cocpt lattice of L)

(2) It does not say that every proper cocpt action should be standard.

---

§ 5 Sharp actions and Anosov representations

Def^{n} (Kassel - Kobayashi and Kassel - Thologan, with a modification) :

G : red, Γ : a fin. gen. gp. $|\cdot|_\Gamma : \Gamma \to \mathbb{N}$ word length.

$S \subset \mathfrak{a}/W$: a closed subset

$j : \Gamma \longrightarrow G$ is a sharp embedding w.r.t. S

$\overset{\text{def.}}{\Longleftrightarrow}$ $\exists\ \varepsilon, M > 0$, $\forall\ \gamma \in \Gamma$.

$$d_{\mathfrak{a}/W}\left(\mu(j(\gamma)), S \right) \geq \varepsilon \cdot |\gamma|_\Gamma - M .$$

' $\mu(\Gamma)$ avoids $\mu(H)$ with a linear speed '

For a closed subgp $H \subset G$,

we say ' w.r.t. H ' instead of ' w.r.t. $\mu(H)$ '

By the properness criterion,

$\Gamma \hookrightarrow G$: a sharp emb. w.r.t. H $\overset{\Rightarrow}{\not\Leftarrow}$ $\Gamma \curvearrowright G/H$: proper.

Rmk : Kassel - Kobayashi introduced sharp actions,

which uses $\varepsilon \cdot \| \mu(j(\gamma)) \| - M$ instead of $\varepsilon \cdot |\gamma|_\Gamma - M$.

$\Gamma \hookrightarrow G$ is a sharp emb.

$\Leftrightarrow$ $\Gamma \curvearrowright G/H$: sharp action and

$\Gamma \hookrightarrow G$ is a quasi - isometric emb.

Ex. : G/H : red, $L \subset G$: a red. subgp, $L \curvearrowright G/H$: proper.

For any discrete subgp Γ of G,

the standard proper action $\Gamma \curvearrowright G/H$ is sharp (in KK's sense),

and is sharply embedded w.r.t. H (in KT's sense) if it is quasi-isometrically embedded.

Th^{m} (Kassel - Tholozan) :

$\Gamma \curvearrowright G/H$: proper. cocpt $\Rightarrow$ $\Gamma \hookrightarrow G$ is a sharp emb. w.r.t. H.

Rmk : This was previously known in some special cases :

- **Kassel** : $G/H = (PSL(2.\mathbb{R}) \times PSL(2.\mathbb{R}))/\Delta PSL(2.\mathbb{R})$
- **Guéritaud - Kassel** :
$$G/H = (PO(n.1) \times PO(n.1))/\Delta PO(n.1) \quad (n \geq 2)$$
- **Guéritaud - Guichard - Kassel - Wienhard** :
$$G/H = (G' \times G')/\Delta G', \quad \mathrm{rk}_{\mathbb{R}} G' = 1.$$

To introduce Anosov representations, let us now assume that our reductive Lie gp G satisfies $W_0 = W$, i.e. W is the 'root-theoretic' Weyl group.

$\rightsquigarrow$ W is a finite Coxeter group generated by the orthogonal reflections w.r.t. the hyperplanes defined by 'simple restricted roots',

↑ do not explain.

Ex. : $G = O(p.q)$ with $p > q$.

The set of simple restricted roots $\Delta = \{\alpha_1. \dots. \alpha_q\}$.

$$\alpha_i \in \mathfrak{a}^*, \qquad \alpha_i \begin{pmatrix} t_1 \\ \vdots \\ t_q \end{pmatrix} := \begin{cases} t_i - t_{i+1} & (1 \leq i \leq q-1) \\ t_i & (i = q) \end{cases}$$

Let $\Delta \subset \mathfrak{a}^*$ be the set of simple restricted roots of G, and put $\mathfrak{a}^+ := \{ X \in \mathfrak{a} \mid {}^\forall \alpha \in \Delta, \ \alpha(X) \geq 0 \}$.

Then, $\mathfrak{a}^+$ is a fundamental domain for $W \curvearrowright \mathfrak{a}$,

i.e. $\mathfrak{a}^+ \hookrightarrow \mathfrak{a}$ induces a homeo. $\mathfrak{a}^+ \xrightarrow{\simeq} \mathfrak{a}/W$.

Ex : $G = O(p.q)$ with $p > q$.

$$\mathfrak{a}^+ = \left\{ \begin{pmatrix} t_1 \\ \vdots \\ t_q \end{pmatrix} \,\middle|\, t_1 \geq t_2 \geq \cdots \geq t_q \geq 0 \right\}.$$

Anosov representations (introduced by Labourie)

… there are around 10 equivalent definitions.

(Guichard - Wienhard, Kapovich - Leeb - Porti, Guéritaud - Guichard - Kassel - Wienhard, Bochi - Portie - Sambarino, …)

A convenient definition (not the original one) for us is :

Def^n : G, Δ : as above, $\Theta \subset \Delta$: non-empty.

$j : \Gamma \to G$ is Θ - Anosov

$\overset{\text{def.}}{\Longleftrightarrow}$ $j : \Gamma \hookrightarrow G$ is a sharp emb. w.r.t. $\mathfrak{a}^+ \cap \bigcup_{\alpha \in \Theta} \ker \alpha$.

(· Relation to proper actions and Anosov rep^{ns} … GGKW
(which is not obvious from the original def^n at all !)
· The equivalence of the above def^n and the original one
… proved first by KLP, alternative proof by BPS
(GGKW considered a similar definition))

Rmk : If $\mathrm{rk}_{\mathbb{R}}\, G = 1$ (and $\Theta = \Delta$),

they are precisely convex cocpt representations.

Fact : G, Δ, Θ : as above. Γ : a fin. gen. gp.

(1) If Γ admits a Θ - Anosov rep^n, then Γ is Gromov - hyperbolic.

(2) Θ - Anosov rep^{ns} form an open subset of $\mathrm{Hom}(\Gamma, G)/G$.

Th^m (Kassel - Thilozan) :

G/H : red, $\mathrm{rk}_{\mathbb{R}}\, G - \mathrm{rk}_{\mathbb{R}}\, H = 1$.

$\Gamma \subset G$: a fin. gen. subgp which is not virtually cyclic.

$\Gamma \hookrightarrow G$: a sharp emb. w.r.t. H

$\Rightarrow \ {}^{\exists}\Theta \subset \Delta$: non-empty. Γ is Θ-Anosov

(Proof is based on Kassel '08).

Thm (Kassel - Tholozan) : G/H : red, $\mathrm{rk}_{\mathbb{R}}\, G - \mathrm{rk}_{\mathbb{R}}\, H = 1$.

(1) If Γ admits a proper cocpt action on G/H,

then Γ is Gromov - hyperbolic.

(2) $j : \Gamma \to G$ for which $\Gamma \overset{\text{via } j}{\curvearrowright} G/H$: proper cocpt

form an open subset of $\mathrm{Hom}(\Gamma, G) / G$.

Idea of pf : (1) Immediate from the previous Thm.

(2) Find an appropriate embedding of G into $GL(n, \mathbb{R})$ so that

$\Gamma \hookrightarrow G$: a sharp emb. w.r.t. H

$\Leftrightarrow$ Γ is $\{\alpha_1\}$ - Anosov in $GL(n, \mathbb{R})$

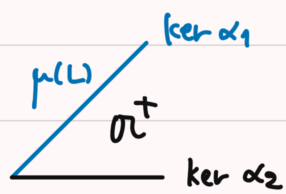
Rmk : • As before, Thm (2) is previously proved by

Kassel / GK / GGKW in special cases.

• Davalo - Riestenberg (essentially) gave an independent proof of Thm (2).

The case $G/H = O(2m, 2) / U(m, 1)$ $(m \geq 2)$ is especially interesting.

$\mathfrak{a} = \mathbb{R}^2$, $\quad \mathfrak{a}^+ = \left\{ \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; t_1 \geq t_2 \geq 0 \right\}$



$\mathfrak{a}_H = \left\{ \begin{pmatrix} t \\ t \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\} = \ker \alpha_1$.

$\rightsquigarrow$ $\Gamma \hookrightarrow G$: a sharp emb. w.r.t. H $\Leftrightarrow$ $\Gamma \subset G$ is $\{\alpha_1\}$ - Anosov

Standard compact quotients (Recall from § 4) :

$L := O(2m, 1) \subset G$, $\quad \Gamma$: a cocpt lattice of L

$\rightsquigarrow$ $\Gamma \curvearrowright G/H$ proper. cocpt

skip

Kassel '12 : For a nicely chosen $\Gamma \subset L$,

$j_0 : \Gamma \hookrightarrow G$ admits a deformation $(j_t : \Gamma \to G)_{t \in \mathbb{R}}$ s.t.

- $j_t(\Gamma)$ is Zariski-dense for $t \neq 0$.
- For $|t|$ suff small, j_t is injective with discrete image, and $j_t(\Gamma) \curvearrowright G/H$: proper. cocpt

Because being $\{\alpha_1\}$-Anosov is an open property!

(Note that this paper is pre-GGKW).

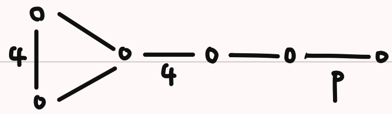
Lee-Marquis, Monclair-Schlenker-Tholozan :
$(4 \leq n \leq 8)$ $\quad$ $({}^\forall n \geq 4)$

Let $G = O(n,2)$ $(n \geq 4)$.

$\exists\, \Gamma \subset G$: $\{\alpha_1\}$-Anosov with $\partial\Gamma \cong S^{n-1}$ s.t.

Γ is not isom. to any lattice of any reductive Lie group.

An example for $n = 4$:

$\Gamma :=$ (the Coxeter group of ) with $4 \leq p < \infty$

(appeared in Sami Douba's mini-course!)

If $n = 2m$, $\Gamma \curvearrowright O(2m,2)/U(m,1)$: proper. cocpt. Thus :

$(\mathrm{vcd}_{\mathbb{Z}}\, \Gamma = \dim S^{2m-1} + 1 = 2m)$

Cor. : $G/H = O(2m,2)/U(m,1)$ with $m \geq 2$.

$\exists\, \Gamma \subset G$: discrete subgp s.t. $\Gamma \curvearrowright G/H$: proper. cocpt

and Γ is not isom. to any lattice of any reductive Lie group.

(This does not require Kassel-Tholozan.)

§6 Proof of the properness criterion ← skip

Thm (Kobayashi, Benoist) :
G : a red Lie gp. $H, L \subset G$: a closed subset.
Then, $H \pitchfork L$ in G $\iff$ $\underset{\uparrow}{\mu(H)} \pitchfork \underset{\uparrow}{\mu(L)}$ in $\mathfrak{a}$

regard as W-inv. subsets of $\mathfrak{a}$

Key Prop. (Kassel '08) : $g, g' \in G$.
(1) $d_{\mathfrak{a}/W}(\mu(gg'), \mu(g)) \leq \|\mu(g')\|$
(2) $d_{\mathfrak{a}/W}(\mu(gg'), \mu(g')) \leq \|\mu(g^{-1})\|$ $(= \|\mu(g)\|)$.

Pf of 'Key Prop. ⇒ Thm' :
By KAK-decomposition, $H \sim \exp(\mu(H))$. $L \sim \exp(\mu(L))$.
Hence, $H \pitchfork L$ in G $\iff$ $\exp(\mu(H)) \pitchfork \exp(\mu(L))$ in G
$\Rightarrow$ —————— " —————— in A
(Want to prove '⇐') $\iff$ $\mu(H) \pitchfork \mu(L)$ in $\mathfrak{a}$

Lem : $C \subset G$: a cpt subset, $R := 2 \cdot \max_{c \in C} \|\mu(c)\|$
Then, for any closed subset M of G,
$\mu(CMC^{-1}) \subset \overline{N}_R(\mu(M))$.

Pf of Lem : For $c_1, c_2 \in C$ and $m \in M$, we have
$d_{\mathfrak{a}/W}(\mu(c_1 \cdot m \cdot c_2^{-1}), \mu(m))$
$\leq d_{\mathfrak{a}/W}(\mu(c_1 \cdot m \cdot c_2^{-1}), \mu(c_1 m)) + d_{\mathfrak{a}/W}(\mu(c_1 \cdot m), \mu(m))$
$\underset{\uparrow}{\leq} \|\mu(c_2)\| + \|\mu(c_1)\| \leq R$. ▨
(Key Prop.)

Take any compact subset C of G. We have

$$\underbrace{c\cdot \exp(\mu(H))\cdot c^{-1} \cap \exp(\mu(L))}_{(*)} \subset \mu^{-1}\big(\mu(c\cdot\exp(\mu(H))\cdot c^{-1}) \cap \mu(L)\big)$$

$$\overset{\text{(Lem.)}}{\subset} \mu^{-1}\big(\underbrace{\overline{N_R(\mu(H))} \cap \mu(L)}_{\text{cpt !}}\big).$$

Now, since μ is a proper map, $(*)$ is compact. ▨

The rest of this section is devoted to the proof of Key Prop.

Define $\underline{p} : \mathfrak{p} \to \mathfrak{a}/W$ by $p(\underset{K}{\mathrm{Ad}(k)} \cdot \underset{\mathfrak{a}}{X}) := W\cdot X$.

[$\underline{\text{Technical Lem.}}$: $\forall X, Y \in \mathfrak{p}$, $d_{\mathfrak{a}/W}(p(X), p(Y)) \leq \|X - Y\|$.]

$\underline{\text{Pf of 'Tech. Lem} \Rightarrow \text{Key Prop.'}}$:

(1) Write $g = \exp(X_g)\cdot k_g$ $(k_g \in K,\ X_g \in \mathfrak{p})$, and similarly for g, gg'.

$$d_{\mathfrak{a}/W}(\mu(gg'), \mu(g)) = d_{\mathfrak{a}/W}(p(X_{gg'}), p(X_g)) \overset{\text{(Tech. Lem.)}}{\leq} \|X_{gg'} - X_g\|_{\mathfrak{p}}$$

$$\underset{(G/K\ :\ \text{non-pos. curved !})}{\leq} d_{G/K}(\exp(X_{gg'})\cdot p_0, \exp(X_g)\cdot p_0) = d_{G/K}(gg'\cdot p_0, g p_0)$$

$$= d_{G/K}(g'\cdot p_0, p_0) = \|\mu(g')\|.$$

(2) $d_{\mathfrak{a}/W}(\mu(gg'), \mu(g')) = d_{\mathfrak{a}/W}(-\mu(gg'), -\mu(g'))$

$$= d_{\mathfrak{a}/W}(\mu(g'^{-1}\cdot g^{-1}), \mu(g'^{-1}))$$

$$\underset{(1)}{\leq} \|\mu(g^{-1})\|.$$ ▨

To prove Technical Lem., we introduce :

$\underline{\text{Def}^{\text{n}}}$: $X \in \mathfrak{a}$: $\underline{\text{regular}}$

$\overset{\text{def.}}{\iff}$ $\mathfrak{a}$ is the unique max. ab. subsp. of $\mathfrak{p}$ containing X.

Put $\underline{\mathfrak{a}_{\text{reg}}} := \{X \in \mathfrak{a} \mid X \text{ is regular}\}$

One can see that $\mathfrak{a}_{\text{reg}}$ is open dense in $\mathfrak{a}$.

$\left(\mathfrak{a}_{\text{reg}} = \mathfrak{a} \setminus \bigcup_{\lambda\in\Sigma} \ker\lambda, \text{ where } \Sigma \text{ is the rest. root system} \right)$

Pf of Technical Lemma : Conj. by K $\rightsquigarrow$ We may assume $X \in \mathfrak{a}$.

$\mathfrak{a}_{reg}$ is dense $\rightsquigarrow$ We may even assume $X \in \mathfrak{a}_{reg}$.

Lem : Take $k_0 \in K$ at which
$$\begin{array}{ccc} K & \longrightarrow & \mathbb{R} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ k & \longmapsto & B(X, \mathrm{Ad}(k) Y) \end{array}$$
attains a maximum.

Then, $\mathrm{Ad}(k_0) \cdot Y \in \mathfrak{a}$.

Pf of Lem. : For any $Z \in \mathfrak{k}$, we have

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} B(X, \mathrm{Ad}(\exp(tZ) \cdot k_0) Y) \\ &= B\left(X, \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(\exp(tZ)) \cdot \mathrm{Ad}(k_0) Y\right) \\ &= B(X, [Z, \mathrm{Ad}(k_0) Y]) \underset{\substack{\uparrow \\ (B : \mathfrak{g}\text{-inv.})}}{=} -B(Z, [X, \mathrm{Ad}(k_0) Y]) \end{aligned}$$

Since $[\underset{\substack{\cap \\ \mathfrak{p}}}{X}, \underset{\substack{\cap \\ \mathfrak{p}}}{\mathrm{Ad}(k_0) Y}] \in \mathfrak{k}$ and B is neg-def. on $\mathfrak{k}$,

we have $[X, \mathrm{Ad}(k_0) Y] = 0$.

Since $X \in \mathfrak{a}_{reg}$, we conclude that $\mathrm{Ad}(k_0) Y \in \mathfrak{a}$. ▨

Take $k_0 \in K$ as above. Then, $p(Y) = W \cdot \mathrm{Ad}(k_0) Y$ and

$$\begin{aligned} d_{\mathfrak{a}/W}(p(X), p(Y))^2 &= \min_{w \in W} \| X - w \cdot \mathrm{Ad}(k_0) \cdot Y \|^2 \\ &= \|X\|^2 + \|Y\|^2 - 2 \underbrace{\max_{w \in W} B(X, w \cdot \mathrm{Ad}(k_0) \cdot Y)}_{\substack{\shortparallel \\ (*)}}. \end{aligned}$$

Since $(*) \geq B(X, \mathrm{Ad}(k_0) Y) \geq B(X, Y)$,

We have
$$\begin{aligned} d_{\mathfrak{a}/W}(p(X), p(Y))^2 &\leq \|X\|^2 + \|Y\|^2 - 2B(X, Y) \\ &= \|X - Y\|^2. \end{aligned}$$
▨

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§7 : A lecture by Maciej Bocheński.

arXiv : 2501.14274

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§8 Geometric fibration conjecture for proper cocpt actions and sphere bundles

Setting : G/H : red. $\quad \Gamma \subset G$: torsion-free.

$\Gamma \curvearrowright G/H$: proper. cocpt (and automatically free)

$\rightsquigarrow$ We have a double fibration

$$\begin{tikzcd} & \Gamma\backslash G/K_H \arrow[dl, "\pi_\Gamma"'] \arrow[dr, "\rho_\Gamma"] & \\ \Gamma\backslash G/H & & \Gamma\backslash G/K . \end{tikzcd}$$

fibre H/K_H. (contractible) $\qquad$ fibre K/K_H. (cpt)

> **Geometric fibration conjecture** (Tholozan) :
> $\exists\, M_\Gamma \subset \Gamma\backslash G/K$: a submfd s.t.
> π_Γ induces a diffeo. $\rho_\Gamma^{-1}(M_\Gamma) \xrightarrow{\cong} \Gamma\backslash G/H$.

(In other words, $\exists\, \sigma_\Gamma : \Gamma\backslash G/H \longrightarrow \Gamma\backslash G/K_H$ a section of π_Γ whose image is a K/K_H - foliated submfd of $\Gamma\backslash G/K_H$.)

Rmk : (1) This conj. asserts in particular that $\exists$ a smooth fibre bundle $\Gamma\backslash G/H \twoheadrightarrow M_\Gamma$ with total space $\Gamma\backslash G/H$ and typical fibre K/K_H.

(2) M_Γ must be a cpt mfd (w/o boundary) s.t.

- $\pi_1(M_\Gamma) \cong \Gamma$.
- The universal cover of M_Γ is contractible.

Ex : $G/H = \mathbb{H}^{n,1} = PO(n,2)/P(O(n,1)\times O(1))$

(the anti-de Sitter space AdS^{n+1}).

$K = P(O(n)\times O(2))$. $\qquad K_H = P(O(n)\times O(1)\times O(1))$.

$$\Gamma\backslash\mathbb{H}^{n,1} \xleftarrow[\text{fibre } \mathbb{H}^n]{\pi_\Gamma} \begin{pmatrix}\text{based timelike geodesics}\\ \text{in } \Gamma\backslash\mathbb{H}^{n,1}\end{pmatrix} \xrightarrow[\text{fibre } \mathbb{RP}^1]{p_\Gamma} \begin{pmatrix}\text{timelike geodesics}\\ \text{in } \Gamma\backslash\mathbb{H}^{n,1}\end{pmatrix}$$

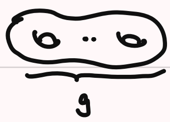
The geometric fibration conjecture says that $\Gamma\backslash\mathbb{H}^{n,1}$ should admit a foliation by timelike geodesics.

<u>Supporting evidences</u> :

(1) The GF Conj. is true for standard proper cocpt actions.

(2) The validity of the GF Conj. is an open property in the moduli space of proper cocpt actions.

(3) (Guéritaud - Kassel) :
The GF conj. is true for $G/H = (PO(n,1)\times PO(n,1))/\Delta PO(n,1)$.
($n=2$ ... anti-de Sitter 3-mfds)

Rmk : Let $n=2$ and $\Gamma_0 := \pi_1\Big($  $\Big)$. $(g \geq 2)$

$\rightsquigarrow$ The moduli space of proper cocpt actions is not connected (Salein '00)
But the GF Conj. is true for every point in the moduli space.

(4) (Monclair - Schlenker - Tholozan + Kassel - Tholozan)
The GF conj. is true for $G/H = O(2n,2)/U(n,1)$

Rmk : As we saw in §5, Γ can be a discrete subgroup not isomorphic to any lattice of any reductive Lie gp in this case.
But the GF Conj. is true even for such Γ.

Prop : Assume the GF Conj is true.

N : the normal bundle of K/K_H in G/H

$(N = K \times_{K_H} \mathfrak{p}/\mathfrak{p}_H)$.

Then, $\boxed{\text{$G/H$ admits a proper cocpt action only when N is a trivial bundle.}}$

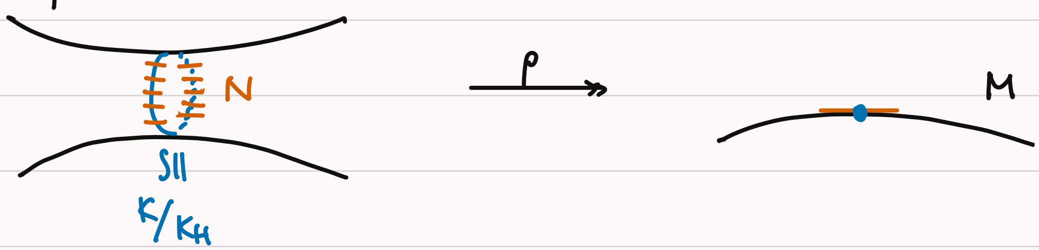
Rmk : This is conjectured by Kobayashi - Yoshino in mid - '00s.

Sketch of pf : If the GF Conj were true, we have

$$
\begin{tikzcd}
\Gamma\backslash G/H & \Gamma\backslash G/K_H \arrow[l, "\pi_\Gamma"'] \arrow[r, "\rho_\Gamma"] & \Gamma\backslash G/K \\
\Gamma\backslash G/H \arrow[u, equal] & \rho_\Gamma^{-1}(M_\Gamma) \arrow[u, phantom, sloped, "\subset"] \arrow[l, "\cong"', "\text{diffeo.}"] \arrow[r, two heads] & \exists\, M_\Gamma \arrow[u, phantom, sloped, "\subset"]
\end{tikzcd}
$$

$$
\rightsquigarrow \quad
\begin{tikzcd}
G/H & G/K_H \arrow[l, "\pi"'] \arrow[r, "\rho"] & G/K \\
G/H \arrow[u, equal] & \rho^{-1}(M) \arrow[u, phantom, sloped, "\subset"] \arrow[l, "\cong"', "\text{diffeo.}"] \arrow[r, two heads] & \exists\, M \arrow[u, phantom, sloped, "\subset"]
\end{tikzcd}
$$

$G/H = \rho^{-1}(M)$



We proved a somewhat weaker result without assuming the GF Conj :

Th^m (Kassel - M. - Tholozan) ← (hope to put on arXiv next month ...)

G/H, N : as before.

$S(N)$: the unit sphere bundle for N

[Then, G/H admits proper cocpt actions only if $S(N)$ has the same fibrewise-homotopy type as the trivial sphere bundle.]

(Fibrewise homotopy … homotopy of total spaces compatible with the projections to the base space.)

Ex : $G/H = \mathcal{H}_{\mathbb{C}}^{2 \cdot 4n, 1} := PU(24n, 2) / P(U(24n, 1) \times U(1))$ $(n \geq 1)$

Then, $S(N)$ is fibrewise-homotopy equiv. to the trivial sphere bundle, but N is not trivial.

Apply Th^m to $G/H = \mathcal{H}^{p,q}$ $\left(:= PO(p, q+1) / P(O(p,q) \times O(1)) \right)$.

$\rightsquigarrow$ $K/K_H = \mathbb{R}P^q$, $N = (\text{tautological line bundle})^{\oplus p}$

[**Fact** (Adams '62) :

In this situation, $S(N)$ has the same fibrewise-homotopy type as the trivial sphere bundle

if and only if p is divisible by $2^{h(q)}$.]

$$h(q) := \begin{cases} \lfloor \frac{q}{2} \rfloor & (q \equiv 0, 6, 7 \mod 8) \\ \lfloor \frac{q}{2} \rfloor + 1 & (q \equiv 1, 2, 3, 4, 5 \mod 8) \end{cases}$$

(cf. Kobayashi-Yoshino : the 'tangential' version)

[Cor. : $\mathcal{H}^{p,q}$ admits proper cocpt actions only if p is divisible by $2^{h(q)}$.]

p	N	$2N$	$4N$	$8N$	$16N$	$32N$	$64N$	$\cdots$
q	0	1	$2, 3$	$4, 5, 6, 7$	8	9	$10, 11$	$\cdots$

*

Previously known results :

$\nexists$ a proper cocpt action on $\mathcal{H}^{p,q}$ if

- $p \geq q \geq 1$ (Calabi-Markus phenomenon)

- p : odd. $q \geq 1$ (Thelozan. M.)

(improve earlier results by Kulkarni and by Benoist)

$\exists$ a proper cocpt action on $\mathcal{H}^{p,q}$ if

$(p,q) = (0,n) \leftarrow$ trivial

$(n,0) \leftarrow$ hyperbolic

$\left.\begin{array}{l}(2n,1)\\(4n,3)\end{array}\right\} \leftarrow$ Kulkarni

$(8,7) \leftarrow$ Kobayashi

(Standard quotients

$L = U(n,1)$

$Sp(n,1)$

$Spin(8,1)$)

Rmk: The smallest unsolved case: $(p,q) = (4,2)$.

In this case, there is a 'subgeometry':

$$G_{2(2)}/SU(2,1) \xrightarrow[\text{diffeo.}]{\cong} O(4,3)/O(4,2) = \hat{\mathcal{H}}^{4,2}.$$

$\uparrow$ $\exists$ a $G_{2(2)}$-inv. almost cpx str., similarly to $S^6 = G_{2,\mathrm{cpt}}/SU(3)$.

We even do not know if $G_{2(2)}/SU(2,1)$ admits proper cocpt actions or not.

Proof sketch of Thm:

$\mathfrak{q} := \mathfrak{h}^{\perp}$ in $\mathfrak{g}$

$$\rightsquigarrow \quad N = K \times_{K_H} (\mathfrak{p} \cap \mathfrak{q}) \longrightarrow G/H \quad \text{is diffeo.}$$

$$[k, X] \longmapsto k \cdot \exp(X) \cdot H \qquad \text{(Kobayashi '89)}$$

$$\begin{tikzcd} & G/K_H \arrow[dl, "\pi"'] \arrow[dr, "\rho"] & \\ G/H \arrow[ur, bend right, "\exists \sigma : \Gamma\text{-equivariant section}"'] & & G/K \end{tikzcd}$$

(since H/K_H is contractible)

$$\begin{tikzcd} & & & G \arrow[d] \\ \mathfrak{p} \cap \mathfrak{q} \arrow[r, hook] \arrow[urrr, "\exists \tilde{\sigma}"] & G/H \arrow[r, "\sigma"] & & G/K_H \end{tikzcd}$$

(since $\mathfrak{p} \cap \mathfrak{q}$ is contractible)

We can take $\tilde{\sigma}$ so that $\tilde{\sigma}(0) = 1$.

Put $\Psi : K/K_H \times (\mathfrak{p} \cap \mathfrak{q}) \longrightarrow G/H \cong N$

$(k, X) \longmapsto \tilde{\sigma}(X) \cdot x$

Ψ is not even compatible with the projections to K/K_H, but satisfies the assumptions of the following (purely homotopy - theoretic) lemma. ▨

← skip

Lem : $X = (X, x_0)$: a connected based finite CW cpx,

$\pi : E \to X$. $\pi' : E' \to X$ vector bundles

Identify their zero - sections with X.

$S(E)$ and $S(E')$ have the same fibrewise - homotopy type if $\exists\ \Psi : E \longrightarrow E'$ s.t. :

(i) $\Psi|_{\text{zero-section}} = \mathrm{id}_X$.

(ii) $\Psi(E_{x_0}) = E'_{x_0}$.

(iii) The induced map $\Psi : E_{x_0} \setminus \{0\} \to E'_{x_0} \setminus \{0\}$ is a homotopy equivalence.

(iv) $\Psi^{-1}(\text{zero - section})$ is a cpt subset of E.

(The cptness of $\Gamma \curvearrowright G/H$ is needed to verify (iv))