The arithmetic of power series, II

On Diophantine effectivization

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November 10, 2025
Centre International de Rencontres Mathématiques in Luminy

Bost's formal-analytic arithmetic surfaces

Background object

A datum $\widetilde{\mathcal{V}} = \left(\widehat{\mathcal{V}}, (V \smallsetminus \mathcal{K}, O), c, \iota\right)$ made of:

- $\widehat{\mathcal{V}}$ a smooth formal curve over $\operatorname{Spec} \mathbf{Z}$ $\widehat{\mathcal{V}} = \operatorname{Spf} \mathbf{Z}[\![X]\!]$
- V a closed Riemann surface with antiholomorphic involution c
- $O \in V(\mathbf{R}) =: V^c$ and $\mathcal{K} \subset V \smallsetminus \{O\}$ a compact with $\mathcal{K}^c = \mathcal{K}$
- $\iota : \widehat{\mathcal{V}}_{\mathbf{C}} \xrightarrow{\simeq} \widehat{V}_O$ an isomorphism of smooth complex formal curves

Actors

- $\mathcal{O}(\widetilde{\mathcal{V}}) = \left\{(\widehat{f}, f^{\mathrm{an}}) \in \mathcal{O}(\widehat{\mathcal{V}}) \times \mathcal{O}(V \smallsetminus \mathcal{K}) \;:\; \iota^* f^{\mathrm{an}} = \widehat{f}_{\mathbf{C}}\right\}$
- $\mathcal{M}(\widetilde{\mathcal{V}})$: *ditto* on replacing both $\mathcal{O}$ rings by $\operatorname{frac} \mathcal{O}$
- $\mathcal{O}^{\mathrm{b}}(\widetilde{\mathcal{V}})$, $\mathcal{M}^{\mathrm{b}}(\widetilde{\mathcal{V}})$: *ditto* by considering the ring of *bounded* holomorphic functions

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An Ansatz for Diophantine approximation

We need to work with *almost functions*, where the gluing discrepancy function $|\iota^* \widehat{f}_{\mathbf{C}} - f^{\mathrm{an}}|$ is small. *Notation:* $\overline{\mathbf{D}}(\rho) := \{|z| \leq \rho\}$, $\overline{\mathbf{D}} := \overline{\mathbf{D}}(1)$

A simple explicit case of the holonomy bounds

Let $\varphi \in \mathcal{O}(\overline{\mathbf{D}})$ with $\varphi(0) = 0$ and $|\varphi'(0)| > 1$, and $\rho \in (0, 1]$. Take $\widehat{f}_1, \ldots, \widehat{f}_m \in \mathbf{Z}[\![X]\!]$, parameters $D \in \mathbf{N}$ and *finite* $L \geq mD$ which is a strict upper bound on all $\mathrm{ord}_{X=0}\left(\sum_{i=1}^m Q_i \widehat{f}_i\right)$ with $\deg Q_i < D$ (not all zero).

Consider decompositions $\widehat{f}_i = f_i^{\mathrm{an}} + h_i$ where $\varphi^* f_i^{\mathrm{an}} \in \mathcal{O}(\overline{\mathbf{D}})$ while $\varphi^* h_i \in \mathcal{O}(\overline{\mathbf{D}}(\rho))$ with

$$\sup_{i, \mathbf{T}(\rho)} |\varphi^* h_i| \leq \rho^L (1 - \rho) \sup_{i, \mathbf{T}} |\varphi^* f_i^{\mathrm{an}}|.$$

Then the number m of our formal functions does not exceed

$$\frac{2}{1 - 1/mD} \frac{\int_{\mathbf{T}} \log^+ |\varphi| \, \mu_{\mathrm{Haar}} + D^{-1} \log \sup_{i, \mathbf{T}} |\varphi^* f_i^{\mathrm{an}}| + D^{-1} \log(4mD)}{\log |\varphi'(0)|}.$$

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Idea of where the Ansatz comes from

Perelli and Zannier's Dynamic Box gives:

$$E_D([0,H)) := \left\{ \sum_{i=1}^{m} Q_i \widehat{f_i} \ : \ Q_i \in \mathbf{Z}[x]_{<D}, \text{ coefficients in } [0,H) \right\},$$

$$H^{mD} = \# E_D([0,H)) \leq \prod_{n=0}^{L-1} (1 + 2A_n),$$

if $|c| \leq A_n$ for all $cx^n + \ldots = F - G \in E_D^{(n)}((-H,H))$.

For this upper bound, we want to use the almost-analytic properties of $\varphi^* F - \varphi^* G = c\varphi'(0)^n + \ldots$.

In this expression we have $\varphi^* Q_i, \varphi^* f_i^{\mathrm{an}} \in \mathcal{O}(\overline{\mathbf{D}})$ with controlled growth on $\mathbf{T} = \partial \overline{\mathbf{D}}$, but we only have $\varphi^* h_i \in \mathcal{O}(\overline{\mathbf{D}}(\rho))$. And so we truncate $\varphi^* h_i = v_{i,n} + O(z^{n+1})$ in order to use Cauchy's or Jensen's integral formulas for holomorphic functions in the disc $\overline{\mathbf{D}}$.

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The Dynamic Box

- $\sup_{\mathbf{T}} |v_{i,n}| \leq \sum_{k=0}^{n} \left| \left[z^k \right] \{ \varphi^* h_i \} \right| \leq \sum_{k=0}^{n} \oint_{\mathbf{T}(\rho)} \left| z^{-k} \varphi^* h_i \right| \mu_{\mathrm{Haar}}$
- By the geometric progression, estimating the integrand by its maximum on the $|z| = \rho$ contour: $\sup_{\mathbf{T}} |v_{i,n}| \leq \frac{\rho^{-n}}{1-\rho} \sup_{\mathbf{T}(\rho)} |\varphi^* h_i|$
- By assumption hence: $\sup_{\mathbf{T}} |v_{i,n}| \leq \sup_{i,\mathbf{T}} |\varphi^* f_i^{\mathrm{an}}| =: B$
- Subharmonicity or Jensen:

$$\begin{aligned} \log |c \varphi'(0)^n| &\leq \int_{\mathbf{T}} \log \left| \sum_{i=1}^{m} \varphi^* Q_i \cdot (\varphi^* f_i^{\mathrm{an}} + v_{i,n}) \right| \mu_{\mathrm{Haar}} \\ &\leq DT(\varphi) + \log(2mDHB). \end{aligned}$$

- Conditioning product: $H^{mD} \leq \prod_{n=0}^{L-1} \left(1 + \frac{4mDH}{|\varphi'(0)|^n} B e^{DT(\varphi)} \right)$
- In fact there are only mD vanishing filtration jumps in the evaluation module E_D: improvement to $H^{mD} \leq \prod_{n=0}^{mD-1} \left(1 + \frac{4mDH}{|\varphi'(0)|^n} B e^{DT(\varphi)} \right)$
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$$\begin{aligned} H^{mD} &\leq \prod_{n=0}^{mD-1} \left(1 + \frac{4mDH}{|\varphi'(0)|^n} B e^{DT(\varphi)} \right) \\ \rightsquigarrow 1 &\leq \prod_{n=0}^{mD-1} \frac{4mD}{|\varphi'(0)|^n} B e^{DT(\varphi)} \\ &= \frac{(4mD)^{mD}}{|\varphi'(0)|^{\binom{mD}{2}}} B^{mD} e^{mD^2 T(\varphi)} \end{aligned}$$

Remark One: $\mathcal{O}(\widetilde{\mathcal{V}}_{\varphi})$ as the $D \to \infty$ limit

$D \to \infty \implies L \to \infty \implies \rho^L(1-\rho) \to 0 \implies h_i = 0$

The $\mathbf{Q}(X)$-linear span dimension of the $f \in \mathbf{Z}[\![X]\!]$ with $\varphi^* f \in \mathcal{O}(\mathbf{D})$ is finite and bounded above by the *holonomy quotient*

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The Bost–Charles holonomy bound

If $(P, g_{V^\circ,O}).(P, g_{V^\circ,P}) = \widehat{\deg} P^* N_{P/\widetilde{\mathcal{V}}} > 0$ and there is $\alpha \in \mathcal{O}(\widetilde{\mathcal{V}}) \smallsetminus \mathbf{Z}$, then

$$\left[\mathcal{M}(\widetilde{\mathcal{V}}) : \mathbf{Q}(\alpha)\right] \leq \frac{\alpha_*(P, g_{V^\circ,O}).\alpha_*(P, g_{V^\circ,O})}{(P, g_{V^\circ,O}).(P, g_{V^\circ,P})}.$$

Here, $P : \mathrm{Spec}\,\mathbf{Z} \to \widehat{\mathcal{V}}$ *be the unique section;* $V^\circ := V \smallsetminus \mathcal{K}$*;* $P^* N_{P/V}$ *is given the capacitary norm;* $\frac{i}{\pi}\partial\overline{\partial}\, g_{V^\circ,O} = \delta_O - \mu_{V^\circ,O}$ *of* $\mathrm{supp}\,\mu_{V^\circ,O} \subseteq \mathcal{K}$.

Remark Two: effective measures of Irrationality

A typical application occurs from the method of Apéry limits. We may have

$$f_i^{\mathrm{an}} = c_i \cdot (B_i(x) - \eta^i A(x)), \quad i = 1, \ldots, m,$$

for instance, from raising a single $H(x) = B_0(x) - \eta A_0(x)$ to powers.

Suppose $|\eta - p/q| < q^{-\mu}$ along an infinite sequence of rational approximations p/q. They induce $|\eta^i - p^i/q^i| \ll_{|\eta|,i} q^{-\mu}$ as well. With $c_i := q^i$ along the sequence we then can set $\widehat{f_i} := q^i B_i(x) - p^i A(x)$, whence

$$h_i = \widehat{f_i} - f_i^{\mathrm{an}} = \left(q^i \eta^i - p^i\right) A(x)$$

is coefficients-wise small.

Now suppose there is a linear differential operator L with say at most regular singularities at $\{0, \alpha, \beta, \infty\}$, such that $L(A) = L(B_i) = 0$, and that each $B_i(x) - \eta^i A(x)$ is regular at the singularity $x = \alpha$ after the analytic continuation along some choice of path γ from 0 to α. Often we can find a conformally large analytic mapping φ turning $\varphi^* f_i^{\mathrm{an}} \in \mathcal{O}(\overline{\mathbf{D}})$.

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- Zero estimates of Shidlovsky, Chudnovsky, ... in functional bad approximability prove in this holonomic situation that the minimalistic $L = mD + C$ works, where C is in practice explicit and small. It is uniform in the ODE deformation type, and treated as a constant here
- We will ensure

$$\sup_{i,\mathbf{T}(\rho)} |\varphi^* h_i| \leq \rho^{mD+C}(1-\rho) \sup_{i,\mathbf{T}} |\varphi^* f_i^{\mathrm{an}}|$$

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Remark Two: effective measures of Irrationality

- If we set $\rho := \min \left\{ \left| \varphi^{-1}\{\alpha, \beta\} \smallsetminus \{0\} \right| \right\}$, we get $\varphi^* h_i \in \mathcal{O}(\overline{\mathbf{D}}(\rho))$
- Zero estimates of Shidlovsky, Chudnovsky, ... in functional bad approximability prove in this holonomic situation that the minimalistic $L = mD + C$ works, where C is in practice explicit and small. It is uniform in the ODE deformation type, and treated as a constant here
- We will ensure

$$\sup_{i, \mathbf{T}(\rho)} |\varphi^* h_i| \leq \rho^{mD+C}(1-\rho) \sup_{i, \mathbf{T}} |\varphi^* f_i^{\mathrm{an}}|$$

  once we choose $D = D(q)$ according to $q^{-\mu} = \rho^{mD+o(D)}$
- As we run $q \to \infty$ along our sequence of rational approximations, we have $D \to \infty$ and we obtain an effective measure of irrationality for η.

An effective irrationality type quotient

In the situation, the holonomy bound

$$\begin{aligned} m \leq & \frac{2}{1-o(1)} \frac{\int_{\mathbf{T}} \log^{+} |\varphi| \, \mu_{\mathrm{Haar}} + D^{-1} \log \sup_{i,\mathbf{T}} |\varphi^{*} f_i^{\mathrm{an}}| + o(1)}{\log |\varphi'(0)|} \\ = & \frac{2}{1-o(1)} \frac{\int_{\mathbf{T}} \log^{+} |\varphi| \, \mu_{\mathrm{Haar}} + D^{-1} m \log q + o(1)}{\log |\varphi'(0)|} \\ = & \frac{2}{1-o(1)} \frac{\int_{\mathbf{T}} \log^{+} |\varphi| \, \mu_{\mathrm{Haar}} + m^{2} \log (1/\rho) / \mu_{\mathrm{eff}}(\eta) + o(1)}{\log |\varphi'(0)|} \end{aligned}$$

turns to:

$$\mu_{\mathrm{eff}}(\eta) \leq \frac{2m \log(1/\rho)}{\log |\varphi'(0)| - \frac{2}{m} \int_{\mathbf{T}} \log^{+} |\varphi| \, \mu_{\mathrm{Haar}}}$$

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$$
\begin{aligned}
m &\leq \frac{2}{1-o(1)} \frac{\int_{\mathbf{T}} \log^+ |\varphi| \, \mu_{\mathrm{Haar}} + D^{-1} \log \sup_{i,\mathbf{T}} |\varphi^* f_i^{\mathrm{an}}| + o(1)}{\log |\varphi'(0)|} \\
&= \frac{2}{1-o(1)} \frac{\int_{\mathbf{T}} \log^+ |\varphi| \, \mu_{\mathrm{Haar}} + D^{-1} m \log q + o(1)}{\log |\varphi'(0)|} \\
&= \frac{2}{1-o(1)} \frac{\int_{\mathbf{T}} \log^+ |\varphi| \, \mu_{\mathrm{Haar}} + m^2 \log(1/\rho)/\mu_{\mathrm{eff}}(\eta) + o(1)}{\log |\varphi'(0)|}
\end{aligned}
$$

turns to:

$$
\begin{aligned}
\mu_{\mathrm{eff}}(\eta) \leq & \frac{2m \log(1/\rho)}{\log |\varphi'(0)| - \frac{2}{m} \int_{\mathbf{T}} \log^+ |\varphi| \, \mu_{\mathrm{Haar}}} \\
& \frac{2(m-1) \log(1/\rho)}{\log |\varphi'(0)| - \frac{1}{m} \iint_{\mathbf{T}^2} \log |\varphi(z) - \varphi(w)| \, \mu_{\mathrm{Haar}}(z) \mu_{\mathrm{Haar}}(w)}
\end{aligned}
$$

Version for denominators

In this talk, our primary focus is the algebraic Apéry limits and their *effective* Diophantine approximation. For transcendental periods such as Apéry's $\zeta(3)$, we will have necessarily some "integration denominators" in $B_i(x)$ as in Siegel's theory of G-functions: type $\sum \frac{x^n}{d_n}\mathbf{Z}$ with $d_1 \mid d_2 \mid \cdots$ having a finite exponential growth rate τ.

Thus is proved the holonomy bound from Lecture I

More generally, the arithmetic holonomy bound with effective measure of irrationality:

$$m \leq \frac{\iint_{\mathbf{T}^2} \log|\varphi(z)-\varphi(w)|\,\mu_{\mathrm{Haar}}(z)\mu_{\mathrm{Haar}}(w)}{\log|\varphi'(0)| - \tau - (m-1)\left(2\mu_{\mathrm{eff}}(\eta)^{-1} - \ldots\right)\log\frac{1}{\rho}},$$

with a positive elision term, conditionally on the positive denominator.

In practice we use $\tau = \tau\left(1, H, \ldots, H^k\right) \leq \tau(H)\cdot\left(\sum_{j=1}^{k} 1/j\right)$, which is in general sharp.

Example of an asymptotic tightness

Theorem (Chudnovsky–André)

For $\varphi \in \mathcal{M}(\mathbf{C})$ entire meromorphic of Hadamard growth order g, the type $\prod_{i=1}^{r}[1, \ldots, b_i n]$ functions analytically resolved by φ (i.e.: $\varphi^* f \in \mathcal{M}(\mathbf{C})$) has $\mathbf{Q}(x)$-transcendence degree $\leq g\tau = g \cdot (b_1 + \ldots + b_r)$.

We are interested to work with cases where this is an equality. For example:

An example of filling-up the asymptotic holonomy bound

On $f(x) = \log(1-x)$ of type $[1, \ldots, n]$ on $\mathbf{P}^1 \setminus \{1, \infty\}$, we can take $\varphi(z) = 1 - e^{-z} \in \mathcal{O}(\mathbf{C})$ of growth order $g = 1$, and for this the holonomy bound is asymptotically tight up-to a constant:

$$k + 1 \leq \frac{\log \sup_{|z|=R} |1 - e^{-z}|}{\log R - \sum_{j=1}^{k} 1/j} \sim \frac{R}{\log(R/k) - \gamma + o_{R\to\infty}(1)}.$$

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For $\varphi \in \mathcal{M}(\mathbf{C})$ entire meromorphic of Hadamard growth order g, the type $\prod_{i=1}^{r}[1, \ldots, b_i n]$ functions analytically resolved by φ (i.e.: $\varphi^* f \in \mathcal{M}(\mathbf{C})$) has $\mathbf{Q}(x)$-transcendence degree $\leq g\tau = g \cdot (b_1 + \ldots + b_r)$.

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A simplest example of an Apéry limit

Specializing the Hermite–Padé approximation formulas for $\log(1-X)$, or directly by considering the canceling $x = 3 - 2\sqrt{2}$ local monodromy signs, one is led to the following $m = 2$ construction with Apéry limit $\eta = \log 2$ on the D_∞ domain $\mathbf{P}^1 \smallsetminus \{3 - 2\sqrt{2}, 3 + 2\sqrt{2}, \infty\} =: \mathbf{P}^1 \smallsetminus \{\alpha, \beta, \infty\}$:

$$\begin{aligned} \frac{1}{\sqrt{1-6x+x^2}} \int_{3-2\sqrt{2}}^{x} \frac{dt}{\sqrt{1-6t+t^2}} &\in \sum_{n=0}^{\infty} \frac{x^n\,\mathbf{Z}}{[1,\ldots,n]} - \frac{\log 2}{2}\,\mathbf{Z}[\![x]\!] \\ &\in \mathcal{O}\left(\mathbf{C} \smallsetminus [3+2\sqrt{2}, \infty)\right). \end{aligned}$$

Usual irrationality type quotient (round disc choice $\varphi(z) = \beta z = (3 + 2\sqrt{2})z$, hence $\rho = |\alpha/\beta|$ and $|\varphi'(0)| = |\beta|$):

$$\mu(\log 2) \leq \frac{\log(|\beta/\alpha|)}{\log|\beta| - \tau} = \frac{\log(|1/\rho|)}{\log|\varphi'(0)| - \tau} \approx 4.6221.$$

A dihedral method

$$
{}_2F_1\left[\begin{matrix}\alpha \;\; \beta\\ \gamma\end{matrix};x\right] :=
$$

$$
\sum_{k=0}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)\cdot\beta(\beta+1)\cdots(\beta+k-1)}{\gamma(\gamma+1)\cdots(\gamma+k-1)}\,\frac{x^k}{k!}
$$

$$
=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tx)^{-\alpha}\,dt
$$

satisfies second-order linear ODE

$$
x(x-1)\frac{d^2}{dx^2}+((\alpha+\beta+1)x-\gamma)\,\frac{d}{dx}+\alpha\beta,
$$

and Gauss's contiguous relations & Kummer's 24 symmetries give:

$$
{}_2F_1\left[\begin{matrix}-\nu-n \;\; -m\\ -m-n\end{matrix};x\right]-(1-x)^{\nu}\cdot{}_2F_1\left[\begin{matrix}\nu-m \;\; -n\\ -m-n\end{matrix};x\right]
$$

$$
=(-1)^m\frac{\binom{n+\nu}{m+n+1}}{\binom{m+n}{m}}\,{}_2F_1\left[\begin{matrix}-\nu+m+1 \;\; n+1\\ m+n+2\end{matrix};x\right]x^{m+n+1}.
$$

A generating function encoding of the binomial Padé forms

Was anybody familiar with the following exceedingly simple generating function identity? *Please tell us more! How to generalize?*

$$\sum_{m,n=0}^{\infty} {}_2F_1\left[\begin{matrix}-\nu-n & -m\\ -m-n\end{matrix};x\right]\binom{m+n}{m}y^m z^n = \frac{(1-xy)^\nu}{1-y-z+xyz}.$$

On the $m=n$ diagonal we have:

$$\begin{aligned}A_\nu(x,y) &:= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\left\{\binom{n+k}{k}\binom{n+\nu}{n-k}(-x)^{n-k}\right\}y^n\\ &= \sum_{n=0}^{\infty} {}_2F_1\left[\begin{matrix}\nu-n & -n\\ -2n\end{matrix};x\right]\binom{2n}{n}y^n,\end{aligned}$$

are algebraic functions in $y \in \mathbf{P}^1 \setminus \left\{\left(\frac{1\pm\sqrt{1-x}}{x}\right)^2, \infty\right\}$ with dihedral D_r (or $D_{r/2}$) symmetry group, and the diagonal Padé generating function is:

$$H_\nu(x,y) := A_{-\nu}(x,y) - (1-x)^\nu A_\nu(x,y).$$

A generating function encoding of the binomial Padé forms

Was anybody familiar with the following exceedingly simple generating function identity?

$$\sum_{m,n=0}^{\infty} {}_2F_1\left[\begin{matrix} -\nu-n & -m \\ -m-n \end{matrix}; x\right] \binom{m+n}{m} y^m z^n = \frac{(1-xy)^\nu}{1-y-z+xyz}.$$

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are algebraic functions in $y \in \mathbf{P}^1 \smallsetminus \left\{ \left(\frac{1\pm\sqrt{1-x}}{x}\right)^2, \infty \right\}$ with dihedral D_r (or $D_{r/2}$) symmetry group, and the diagonal Padé generating function is:

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The classical hypergeometric method

Consider the representative example of the Thue equation $X^r - 2Y^r = c$ with $c \in \mathbf{Z}$ of $|c| \ll_{\epsilon} |X|^{r/2-1-\epsilon}$. Thue (1909) supposed a solution $(X, Y) = (p_0, q_0)$ with a sufficiently big height, say $q_0 > A(r, \epsilon)$. Then $\sqrt[r]{2} \approx p_0/q_0$ is an excellent rational approximation, and by specializing the functional Hermite–Padé form to

$$H_{1/r}(1 - 2q_0^r/p_0^r, y) \in \mathbf{Z}[\![y/r^2]\!] - (q_0/p_0)\sqrt[r]{2}\,\mathbf{Z}[\![y/r^2]\!],$$

comparing convergence rate to denominators, he was able to conclude that all the other solutions $(X, Y) = (p, q)$ satisfy $|Y| < q_0^B$. Here A and B are explicitly quantifiable functions of r and ϵ, but no procedure is given to determine whether or not the Turing machine that sets out to search for a solution p_0/q_0 with $q_0 > A(r, \epsilon)$ shall eventually halt or run indefinitely.

Thue's insight was that either alternative leads to finiteness of the solution set. It was Alan Baker in the late 1960s who first eventually succeeded to prove the decidability of the Thue equation, but also by the same token that the Thue lever p_0/q_0 in general does not exist. A more algebraic solution was found by Bombieri in 1993 with his equivariant Thue–Siegel method.

The classical hypergeometric method: an example

- $p_0/q_0 := 5/4 \approx \sqrt[3]{2} = 1.259921\ldots$ turns out to work as a Thue lever; the numerology at once gave rise to the "first-ever" explicit sub-Liouville bound $|\sqrt[3]{2} - p/q| > 10^{-6} q^{-2.955}$ (Baker, 1964).
- Just specialize $X := 3/128$ in the functional Hermite–Padé form for $(1-X)^{-1/3}$. Dihedral singularities $\{\alpha, \beta\} = \left(\frac{1 \mp \sqrt{1-X}}{X}\right)^2 \approx \{0.25973, 7196.19\}$ and usual irrationality type quotient

$$\mu_{\mathrm{eff}}\left(\sqrt[3]{2}\right) \le \frac{\log|\beta/\alpha|}{\log|\beta| - 7\log 2 - \frac{1}{2}\log 3} \approx 2.93951 < 3.$$

- Bennett's clean refinement: $|\sqrt[3]{2} - p/q| > q^{-2.5}/4$.
- For general cubic irrationalities α, Thue's argument gave $|\alpha - p/q| > c(\alpha) q^{-2.501}$, but for all we know today even the effectivization of $c(\alpha)$ could in principle be an undecidable problem.

A new continuation based on multivalent holonomy bounds

Our diagonal Hermite–Padé y-generating functions $A_\nu(x,y)$ and $B_\nu(x,y)$ have algebraic degree $\in \{r, r/2\}$, holonomic rank 2, and ramification type $(2,2,r)$ at their three respective singularities:

$$\alpha := \left((1-\sqrt{1-x})/x\right)^2; \quad \beta := \left((1+\sqrt{1-x})/x\right)^2; \quad \infty.$$

Their particular linear combination $H_\nu(x,y)$ overconverges at the singularity $y=\alpha$. These properties imply:

- $\psi_{\alpha,\beta}(z) = \frac{\alpha+\beta}{2}\cdot\left(1-\cosh\left(\frac{z}{\sqrt{\alpha\beta}}\right)\right) + \sqrt{\alpha\beta}\cdot\sinh\left(\frac{z}{\sqrt{\alpha\beta}}\right) \in \mathcal{O}(\mathbf{C})$ has order $g=1$ and $\psi^*A, \psi^*B \in \mathcal{O}(\mathbf{C})$
- $\varphi_{\alpha,\beta}(z) = \alpha\left(1-\sinh^2\left(\sqrt{u(u-z\sqrt{|\beta/\alpha|})}\right)/\sinh^2(u)\right)$ with $\tanh^2(u) := \alpha/\beta$, of order $g=1/2$ with growth $|\alpha|e^{\sqrt{|u|\sqrt{|\beta/\alpha|}|z|}}$ as soon as $|z| \gg |u|$, and $\varphi^*H \in \mathcal{O}(\mathbf{C})$

A dihedral method: origin of the maps $\psi_{\alpha,\beta}$ and $\varphi_{\alpha,\beta}$

Consider more generally $H = B - \eta\, A$ where A and B are holonomic on $\mathbf{P}^1 \smallsetminus \{\alpha, \beta, \infty\}$ with order 2 local monodromies at α and β. We can use two types of degree-2 ramified coverings:

- $\mathbf{P}^1 \to \mathbf{P}^1$ ramified over $\{\alpha, \beta\}$ and taking $\{1, \infty\}$ to ∞
- $\mathbf{P}^1 \to \mathbf{P}^1$ ramified over $\{\beta, \infty\}$.

The first map converts the $\varphi(z) = 1 - e^{-z}$ from the $f(x) = \log(1-x)$ "filling up" example to supply an order-1 map $\psi_{\alpha,\beta} \in \mathcal{O}(\mathbf{C})$ resolving both $\psi^*_{\alpha,\beta} A, \psi^*_{\alpha,\beta} B \in \mathcal{O}(\mathbf{C})$.

The second map resolves β and replaces $\{\alpha, \beta\}$ by

$$\varphi^{-1}\{\alpha\} =: \{\widetilde{\alpha}, \widetilde{\beta}\} = 2\beta \cdot \left(1 \mp \sqrt{1 - \alpha/\beta}\right).$$

Iterating it indefinitely we arrive at our universal explicit order-1/2 map $\varphi_{\alpha,\beta} \in \mathcal{O}(\mathbf{C})$ resolving the Apéry overconvergent combination:

$$\varphi^*_{\alpha,\beta} H = \varphi^*_{\alpha,\beta} B - \eta\, \varphi^*_{\alpha,\beta} A \in \mathcal{O}(\mathbf{C}).$$

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$$\varphi_{\alpha,\beta}^* H = \varphi_{\alpha,\beta}^* B - \eta\, \varphi_{\alpha,\beta}^* A \in \mathcal{M}(\mathbf{C}).$$

Main example: high order roots from a given $\mathbf{G}_m(\overline{\mathbf{Q}})$ point

Select $\varphi(z) := \varphi_{3-2\sqrt{2},3+2\sqrt{2}}(Rz)$ with R an arbitrary radius parameter.

Total nonarchimedean inverse radius $\prod_p R_p^{-1} = \prod_{p|r} R_p^{-1} = r \prod_{p|r} p^{\frac{1}{p-1}}$.
Result:

$$\mu_{\mathrm{eff}}\left(\sqrt[r]{2}\right) \leq \frac{k \log\left(1/\rho\right)}{\log R + \sum_p \log R_p - \frac{\sqrt{R}}{k+1}}$$
$$\ll \frac{k \log R}{\log R - \log\left(r \log r\right) - \frac{\sqrt{R}}{k+1}}.$$

Choice

$$R := c_1 \cdot r \log r, \quad k := c_2 \cdot \sqrt{r \log r}$$

yields an explicit

$$\mu_{\mathrm{eff}}\left(\sqrt[r]{2}\right) \ll \sqrt{r (\log r)^3}.$$

Baker, Waldschmidt, Bombieri on effective Diophantine approximation by finitely generated subgroups of $\mathbf{G}_m$

Generalizing this example leads to a new effective proof of:

Theorem

Consider the following data:

- *a number field* K*;*
- *a place* v *of* K*;*
- *a finitely generated subgroup* $\Gamma < K^\times$ *of the multiplicative group* $\mathbb{G}_m(K) = K^\times$*;*
- *a positive number* $\varepsilon > 0$*.*

From these data, one can extract an effectively computable function $C(K, v, \Gamma, \varepsilon) \in \mathbf{R}$ *such that* $\forall A \in K^\times$*, all solutions* $\gamma \in \Gamma$ *of*

$$|1 - A\gamma|_v \leq H(\gamma)^{-\varepsilon}$$

satisfy the effective height bound $h(\gamma) \leq C(K, v, \Gamma, \varepsilon)\,(1 + h(A))$*.*

Two logarithms: state of art and conjecturally a bit beyond

- The full strength of the Main Diophantine theorem, which is a statement on $n = \mathrm{rank}(\Gamma)$ logarithms in the rational exponents case, derives (Bombieri, 1993; Bilu–Bugeaud, 2000) by a simple Dirichlet approximation argument from Waldschmidt's state-of-art form of the lower bounds in two logarithms in the rational exponents case: $-\log|\sqrt[r]{a} - b|_v \ll_{K,v} h'(a) \cdot \max(1, \log(r/h'(a)))$ $\qquad h' := \max(1, h)$
- Really any explicit bound $-\log|\sqrt[r]{a} - b| \ll r/g_{K,v}(r/h'(a))$ with $g_{K,v}(t) \to \infty$ is sufficient for this task. For instance Laurent's extrapolation determinants (1994) gave more easily $g(t) \sim t/(\log t)^2$ for archimedean v (Laurent, Mignotte, Nesterenko, 1995)
- To restate Waldschmidt's bound in the rational case $K = \mathbf{Q}$: $-\log|b_1 \log n_1 - b_2 \log n_2| \ll \log|2n_1| \cdot \log|2n_2| \cdot \log\left\{1 + \frac{|b_1|}{\log|2n_2|} + \frac{|b_2|}{\log|2n_1|}\right\}$
- Still open in the inhomogeneous case: $-\log|b_1 \log n - b_2| \ll \log|2n| \cdot \log\left\{1 + |b_1| + \frac{|b_2|}{\log|2n|}\right\}$? Baker's theorem: $-\log|b_1 \log n - b_2| \ll \log|2n| \cdot \log\{1 + |b_1| + |b_2|\}$

One logarithm: state of art and now provably a bit beyond

- From Baker's bound $-\log|b_1 \log n - b_2| \ll \log|2n| \cdot \log\{1 + |b_1| + |b_2|\}$, we have in particular
$$\mu(\log n) \ll \log n, \qquad \forall n \in \mathbf{Z}_{\geq 2}.$$
- While we do not (as yet) succeed to prove the inhomogeneous Waldschmidt feature (Mahler's conjecture: $\|\log n\| > n^{-O(1)}$), the $\nu \to 0$ limiting case of the dihedral method with the inclusion of both ($[1, \ldots, n]^2$ type, but algebraically independent) functions $H(y)$ and $E(y) := \int_0^y H(t)\, dt$ leads routinely to this improvement of Baker's record on the irrationality measures of logarithms:
$$\mu(\log n) \ll \sqrt{\log n} \cdot \log\log n, \qquad \forall n \in \mathbf{Z}_{\geq 3}.$$
Indeed
$$\mu(\log q) \ll \sqrt{1 + h(q)} \cdot \log(2 + h(q)), \quad \forall q \in \mathbf{Q} \setminus \{0, 1\}.$$

The upper-triangular $[n/m]$ Padé table of $\log(1-x)$

From

$$\sum_{m,n=0}^{\infty} {}_2F_1\left[\begin{matrix}-\nu-n & -m \\ -m-n & \end{matrix};x\right]\binom{n+m}{m}y^m z^n = \frac{(1-xy)^\nu}{1-y-z+xyz}$$

the first two $\nu = 0$ Taylor series terms give the $y^m z^n, n \geq m$ double generating function of the Hermite–Padé forms for $\log(1-x)$ as simply

$$\frac{\log(1-xy)+\log(1-xz)}{1-y-z+xyz} - \frac{\log(1-x)}{1-y-z+xyz}.$$

Thank you for you attention!